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**COMPUTATIONAL OPTIMAL CONTROL OF NONLINEAR
SYSTEMS WITH PARAMETER UNCERTAINTY**

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Abstract

Computational Optimal Control of Nonlinear Systems with Parameter

Uncertainty

by

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A number of emerging applications in the field of optimal control theory require the computation of an open-loop control for a dynamical system with uncertain parameters. In this dissertation we examine a class of *uncertain optimal control problems*, in which the goal is to minimize the expectation of a predetermined cost functional subject to such an uncertain system. We provide a computational framework for this class of problems based on a discretization of the parameter space. In this approach, a set of nodes from the parameter space and corresponding weights are selected, and the expectation of the cost functional is approximated by a finite sum. This process results in a sequence of standard optimal control problems which can be solved using existing techniques. However, it is well-known that an inappropriately designed discretization scheme for an optimal control problem may fail to converge to the optimal solution, therefore further analysis must be performed to examine the convergence properties of the scheme. We provide this analysis for a scheme based on quadrature methods for the approximation of the expectation in the cost functional. This analysis demonstrates that an accumulation point of a sequence of optimal solutions to the approximate problem is an optimal solution of the original problem. Furthermore, we examine the convergence

of the adjoint states for the approximation based on the quadrature scheme, which leads to a Pontryagin-like necessary condition which must be satisfied by these accumulation points. To address the exponential growth of computational cost with respect to the dimension of the parameters, we introduce a numerical algorithm based on sample average approximations, in which an independently drawn random sample is taken from the parameter space, and the expectation in the objective functional is approximated by the sample mean. Using a generalization of the strong law of large numbers, we analyze the convergence properties of this approximation. In addition, we develop an optimality function for the class of uncertain optimal control problems based on the $\mathcal{L}_2$-Frechet derivative of the objective functional, which provides a necessary condition for an optimal solution. By demonstrating that an accumulation point of a sequence of stationary points for the approximate problem is a stationary point of the original problem, we demonstrate the approximation scheme based on sample averages is consistent in the sense of Polak. These numerical algorithms for the uncertain optimal control problem are applied to real-world scenarios from the fields of optimal search theory and ensemble control.

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Chapter 1

Introduction

The field of optimal control has a long and storied history. In 1696 Johann Bernoulli proposed the brachistochrone problem in which the goal is to determine the curve which will minimize the time needed for an object to travel between two points [86]. This drew the interest of several giants of mathematics, including Leibniz, l'Hopital, and Newton, who all proposed solutions. This gave rise to the field of *variational analysis*, which is concerned with minimization problems in which the decision variable is a function, e.g., the problem of finding the shortest path between two points or the smallest area which can be enclosed with a specified perimeter.

Over the past century, a large research effort has been put into *optimal control* problems, which differ from standard calculus of variation problems in that the function must additionally satisfy a specific dynamic constraint. In this thesis, we focus on optimal control problems with nonlinear dynamics and pointwise control constraints, for example, the following optimal control problem with Bolza cost.

Bolza Problem. Find a state-control pair (x, u) to minimize the objective functional

$$J(x, u) = F(x(T)) + \int_0^T r(x(t), u(t))dt, \qquad (1.1)$$

subject to the dynamical system

$$\dot{x}(t) = f(x(t), u(t)), \qquad\qquad x(0) = x_0, \qquad (1.2)$$

and the control constraint

$$g(u(t)) \leq 0 \text{ for all } t \in [0, T]. \qquad (1.3)$$

Here $x : [0, T] \mapsto \mathbb{R}^{n_x}, u : [0, T] \mapsto \mathbb{R}^{n_u}, F : \mathbb{R}^{n_x} \to \mathbb{R}, r : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$, and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_x}$.

While these problems do not typically lend themselves to direct solutions, the celebrated necessary condition known as Pontryagin's minimum principle provides a Hamiltonian minimization criterion which often leads to a solution [63]. The principle provides a boundary value problem known as the state-adjoint system. The solutions to this boundary value problem serve as candidate solutions to the nonlinear optimal control problem. In some special cases, this approach provides an analytic solution, as in the case when the problem has linear dynamics and a quadratic cost functional. However, it is well known that in general, problems with nonlinear dynamics and control constraints are not guaranteed to admit a closed form solution. Even in these cases, Pontryagin's minimum principle provides a tool to solve the control problem by solving the boundary value problem numerically.

In the class of algorithms for optimal control known as *indirect methods*, the approach is to apply existing methods for the numerical solution of differential equations to the boundary value problem formulated using Pontryagin's minimum principle. However, indirect methods for nonlinear optimal control have two major shortcomings. First, formulating the boundary value problem can be very labor intensive and must be performed by an expert in the field of optimal control. Second, it is well known that the boundary value problem resulting from Pontryagin's minimum principle is very sensitive to the initial guess. It makes a good initial guess necessary for a boundary value problem solver to converge. For many engineering applications, finding a good initial guess, especially for adjoint variables, is very challenging, since these adjoint variables typically do not have clear physical meaning.

In recent decades, a new class of algorithms called *direct methods* have been developed which do not suffer from these drawbacks. In this approach, a discretization scheme is applied in the time domain, resulting in a sequence of approximating high-dimensional nonlinear programming problems which can be solved using existing numerical optimization techniques. However, the apparent simplicity of this approach belies deep theoretical issues encountered when approximating optimization problems. For each proposed discretization scheme, careful analysis must be performed to guarantee that the scheme provides a meaningful approximation to the optimal control problem. Indeed, counterexamples show that an inappropriately designed discretization scheme will lead to incorrect results when applied to control problems [14]. Therefore it is important to demonstrate *consistency* of the discretization scheme, a property

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which guarantees that accumulation points of a sequence of optimal solutions to the approximate problem will be optimal solutions to the original problem. Such consistency results have been demonstrated for a number of different discretization schemes, including Euler [61], Runge-Kutta [31] and Pseudospectral [29].

Together, theoretical results such as Pontryagin's minimum principle and consistency results for direct computational methods comprise a framework in which nonlinear optimal control problems can be solved numerically. In this framework, the optimal control problem is approximated using a discretization in the time domain, and the resulting approximate problem is solved using existing numerical optimization methods. Once consistency of the discretization scheme has been demonstrated, the optimal solution to the approximate problem is known to provide a reasonable approximation of the optimal solution to the original nonlinear optimal control problem. This numerically computed control can then be tested against the Pontryagin minimum principle, and controls not satisfying this necessary condition can be disregarded, as they cannot be an optimal solution to the control problem. In this sense, Pontryagin's minimum principle provides a method for the verification and validation of a numerically computed solution.

Most of the current computational nonlinear optimal control methods do not directly address a critical issue in control system design: the appearance of the uncertainty. Since the uncertainty is inherent to every dynamical model, lack of ability to incorporate uncertainty limits the application of computational optimal control in engineering applications. In this thesis we aim to overcome this shortcomings. The

uncertainty can stem from imprecise measurements of physical properties of the system being controlled, incomplete information about the environment, or inability to predict exactly the behavior of another agent. We focus on a class of problems we refer to as the *uncertain optimal control problem*, in which the goal is to minimize the expected cost, given a prior probability distribution for these unknown parameters. This formulation allows optimal control problems which incorporate parameter uncertainty in the intial state, agent dynamics, or the objective functional. The problem is formulated as follows:

Problem C. Determine the control function $u \in L_\infty([0,1]; \mathbb{R}^{n_u})$ that minimizes the cost functional

$$J[x,u] = \int_\Omega \left[F\left(x(1,\omega),\omega\right) + \int_0^1 r(x(t,\omega),u(t),t,\omega)dt \right] p(\omega)d\omega \qquad (1.4)$$

subject to the dynamics

$$\dot{x}(t,\omega) = f(x(t,\omega),u(t),\omega), \qquad (1.5)$$

initial condition $x(0,\omega) = x_0(\omega)$, and the control constraint $g(u(t)) \leq 0$ for all $t \in [0,1]$. In Problem C, Ω is a space of stochastic parameters with $p : \Omega \mapsto \mathbb{R}$ a probability density function, $L_\infty([0,1]; \mathbb{R}^{n_u})$ is the set of all essentially bounded functions, $x : [0,1] \times \Omega \mapsto \mathbb{R}^{n_x}$ and $r : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^1 \times \mathbb{R}^{n_\omega} \mapsto \mathbb{R}$.

In this thesis we provide computational framework for solving this class of uncertain optimal control problems. The computational methods are based on a discretization of the parameter space using either a quadrature or Monte Carlo integration scheme. A finite number of nodes are selected in the parameter space and the state vec-

tor for the uncertain dynamical system is approximated as a large number of decoupled systems. In the literature, other techniques have been used to make a similar approximation. The stochastic collocation [6] method is similar to the quadrature scheme in that a finite number of nodes are selected from the parameter space and the dynamical system is approximated by a tensor product of polynomials. The polynomial chaos method approximates the random state vector by using a Galerkin projection onto a set of orthogonal basis vectors [88]. Both the stochastic collocation and polynomial chaos methods have been applied in the area of uncertainty quantification, in which the main goal is to analyze the propagation of uncertain through dynamical systems, in contrast to the mitigation of uncertainty using optimal control as studied in this thesis.

Recently, the polynomial chaos approach has also been used in an optimal control setting with parameter uncertainty for special cases of Problem C [20, 21, 38, 39]. In these results, a polynomial chaos expansion is used to approximate state and/or control trajectories; then the dynamics are discretized using a Galerkin projection. While the performance of such polynomial chaos based discretization methods were tested through numerical simulations on some simple optimal control problems, e.g., linear control systems with quadratic cost [20], Van der Pol Oscillator [21], there is no rigorous analysis on the consistency and convergence of such schemes for solving uncertain optimal control problems. Similarly, Ref. [16] suggests a method in which the uncertain problem is approximated using a quadratic Taylor expansion of the objective functional and the results are compared to a Monte Carlo simulation, but no analysis of the consistency properties of the method is provided. Some of this consistency analysis has been per-

formed for the related optimal search [22, 23] and ensemble control [73, 74, 76] problems. These results are reviewed in Sections 1.1 and 1.2.

In this thesis, our goal is to provide computational framework, with rigorous consistency and convergence analysis, for a general class of uncertainty optimal control problems represented by Problem C. Our interest in this problem setting has two major motivations. First, this framework can be applied to previously considered control problems, allowing the designers to incorporate uncertainty into the formulation of an optimal control problem. For instance, the uncertain optimal control setting may be used to extend trajectory optimization problems to scenarios which include uncertainty about physical parameters in the dynamical model, characteristics of the environment, or the behavior of other agents. Second, recent applications of computational optimal control have brought to light problems which necessitate incorporation of uncertainty into the problem formulation. In Sections 1.1 and 1.2 we detail two of these applications, optimal search and ensemble control, and demonstrate how they can be formulated as Problem C.

1.1 Optimal Search Theory

The problem of formulating an optimal search strategy for an agent attempting to detect a moving target is one topic which has generated recent interest in this class of uncertain optimal control problems. The field of search theory traces its origins to the U.S. Navy's Antisubmarine Operations Research Group in 1942. Koopman joined

in 1943 and introduced many important concepts and the basic formulation of the problem [43]. A mathematical formulation of optimal search problem frequently used today was provided in the seminal text of Stone [84]. This formulation of the search problem has three main components: a model for the unknown location or motion of the target, a model for the effectiveness of the search effort for a given strategy, and a model for computing the probability of detecting the target when the searchers adhere to a given search plan. Mangel [53] provides a review of the components of the problem and various models used.

Work on the optimal search problem can be divided depending on the model chosen for the motion of the target. The target is usually modelled as either a Markov process (such as Brownian motion or a random walk), or as conditionally deterministic, meaning the motion of the target depends on a random vector whose true value is unknown to the searcher. Work on the problem of Markovian motion first focused on calculating the posterior distribution of the target's position [34, 51, 52], and developing necessary and sufficient conditions for the search plan to be optimal [36, 37, 77]. Ohsumi [55] provides a necessary condition, as well as a method to numerically calculate an optimal search trajectory. Early studies into the conditionally deterministic problem considered in this work focused on target motion subject to additional special restrictions [82, 83] or targets moving in discrete space [64]. The work of Ref. [64] is significant as it is the first to develop a necessary condition for optimality in an optimal control setting. In Ref. [49], an optimal control approach is used to derive a necessary condition in the case of a target moving in continuous space, when the searcher's dynamics are given by

a single integrator with box constraints.

To briefly demonstrate how the search for moving targets can be modeled as an uncertain optimal control problem, consider the problem of a searcher looking for a moving target in order to maximize the probability of detecting the target over some time horizon $[0, T]$. Let the searcher trajectory, $x(t)$, be determined by the dynamical system $\dot{x} = f(x, u)$, with initial condition x_0 and control constraint $g(u(t)) \leq 0$ for all $t \in [0, T]$. Because our model incorporates nonlinear dynamics and control constraint, it can be applied to problems with various types of vehicles, such as autonomous helicopters or surface vessels. However, in the simulations presented in later part of this thesis (see Section 3.5), we assume the dynamics of the search vehicle is modelled by the Dubin's vehicle given by

$$\dot{x}_1(t) = v \cos x_3(t),$$

$$\dot{x}_2(t) = v \sin x_3(t),$$

$$\dot{x}_3(t) = u(t), \quad |u| \leq u_{max}$$

where (x_1, x_2) represents the position of the searcher and x_3 is the heading angle. The forward velocity, v, is a given constant. The control, u, is the turning rate with maximum angular velocity u_{max}.

Now consider a moving target whose motion is assumed to be conditionally deterministic. By conditionally deterministic we mean that the motion of the target depends on a stochastic parameter, and if the value of this parameter were known the location of the target would be known for all time. In other words, there exists a random

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vector $\omega \in \Omega \subset \mathbb{R}^{n_\omega}$, such that the trajectory of the target conditioned on ω is given by $y(\cdot, \omega)$. It is assumed that the probability density of ω over Ω is known to the searchers and is given by $p : \Omega \mapsto \mathbb{R}^+$.

The final component of the search model is a function describing the effectiveness of the searcher. Let $\tilde{r} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \mapsto \mathbb{R}$ be the instantaneous rate of detection such that the probability of detection in a sufficiently small interval $[t, t + \Delta t)$, conditioned on ω, is given by $\tilde{r}(x(t), y(t, \omega))\Delta t$. The rate function $\tilde{r}$ is chosen to model the qualities of sensors such as acoustic and sonar sensors. For example, for sonar type of sensors, the detection rate function can be given by the Poisson scan model

$$\tilde{r}(x(t), y(t, \omega)) = \beta \, \Phi \Big(\frac{F^k - D \, \|x(t) - y(t, \omega)\|^2 - b}{\sigma} \Big),$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, $\|x(t) - y(t, \omega)\|$ is the Euclidean distance between the searcher and the target, β is the scan opportunity rate, F^k is the so-called "figure of merit" (a sonar characteristic), and σ reflects the variability in the "signal excess".

Denote by $P(t)$ the probability of non-detection at time instance t conditioned on ω. Then

$$P(t + \Delta t) = P(t)(1 - \tilde{r}(x(t), y(t, \omega))\Delta t).$$

As $\Delta t \to 0$ we get

$$P(t) = \exp \Big(- \int_0^t \tilde{r}(x(\tau), y(\tau, \omega))dt \Big).$$

Thus the probability that the target is not detected in the time interval $[0, T]$ is given

by the integral

$$J = \int_\Omega \exp\Big(-\int_0^T \tilde{r}(x(t), y(t, \omega)) dt \Big) p(\omega) d\omega. \qquad (1.6)$$

If the goal is to find the control input which produces a search trajectory to maximize the probability of finding the target, the optimal search problem can be formulated as an optimal control problem subject to parameter uncertainty as following: Find a state-and-control pair (x_1, x_2, x_3, u) which minimizes the objective functional:

$$J = \int_\Omega \exp\Big(-\int_0^T \tilde{r}(x(t), y(t, \omega)) dt \Big) p(\omega) d\omega,$$

subject to the dynamics

$$\dot{x}_1(t) = v \cos x_3(t),$$

$$\dot{x}_2(t) = v \sin x_3(t),$$

$$\dot{x}_3(t) = u(t), \quad |u| \leq u_{max}$$

$$x(0) = x_0.$$

Here $v, u_{max} \in \mathbb{R}^+$ and $\tilde{r} : \mathbb{R}^3 \times \mathbb{R}^2 \mapsto \mathbb{R}^+$.

It can be seen that above optimal search problem is indeed a special case of the uncertainty optimal control Problem C, with uncertainty appearing only in the cost functional.

Recent advances in computational power have made it possible to determine the optimal search strategy numerically by solving this optimal control problem. Recent works in optimal search focus on this aspect of the problem, providing numerical algorithms to compute the optimal search strategy in special cases. Ref. [13] provides a

numerical method based on an Euler quadrature scheme for the special case of a target moving at constant velocity in a channel. Ref. [22] uses a composite-Simpson scheme for a problem with more general dynamics and an exponential detection function. In addition, it provides a necessary condition for optimality and analyzes the consistency properties of this method using the approach of Ref. [61, Chapter 3.3]. Ref. [71, 78] provide numerical methods for problems which are discretized in both time and space. In this thesis we extend these results to the broader class of uncertain optimal control problems, as well as providing consistency results for a number of approximation schemes not previously considered in the context of optimal search.

1.2   Ensemble Control

A number of emerging applications in control theory require the design of a single open loop signal to simultaneously control a large number of structurally identical systems (an ensemble) with variance in the system parameters. The variance in system parameters may arise from possibly complex interactions between the systems [28] or inhomogeneity of the control signal due to physical limitations of the equipment [44,45]. This problem first arose in NMR spectroscopy and MRI, where a control law must be developed to prepare the ensemble of nuclear spins in a specific configuration for an experiment. Even a small variance in system parameters can cause dispersion of the controlled state, making the system difficult to control. Over the years, a number of ad hoc algorithms have been developed to design sequences of pulses which compensate for

this dispersion [44,45]. The emerging field of *Ensemble Control* looks to provide a unify-

ing mathematical framework to analyze such problems and provide new computational

tools which will allow extension of this framework to applications. The ensemble is

represented in this framework as a continuum of structurally identical systems indexed

by a parameter (or parameters) which governs the dynamics of the individual systems.

For example, consider the problem of trying to regulate an ensemble of har-

monic oscillators with variation in the natural frequency. This ensemble is governed by

the system

$$
\begin{bmatrix} \dot{x}_1(t,\omega) \\ \dot{x}_2(t,\omega) \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x_1(t,\omega) \\ x_2(t,\omega) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0,\omega) \\ x_2(0,\omega) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (1.7)
$$

for each $\omega \in [0, 20]$, $t \in [0, T]$, and the goal is to find a control u such that $x(T, \omega) = 0$

for all $\omega \in [0, 20]$. The difficulty in this and other ensemble control problems is that

a single control must be used to stabilize all members of the ensemble, and a control

which stabilizes a single member with a specific parameter value may not stabilize

another member with a different parameter value. To see this, consider Figures 1.1,

which show the dispersion of states for a control computed using a Lyapunov method

for the median case ($\omega = 10$) and the worst case ($\omega = 20$) scenario.

The difficulty of stabilizing this ensemble of harmonic oscillators as well as in

designing compensating pulses for magnetic resonance experiments raises the question of

ensemble controllability: whether it is possible to design a single open loop control signal

which will simultaneously transfer all members of the ensemble from a given state to a

neighborhood of a desired final state. Here both the initial and final states may depend

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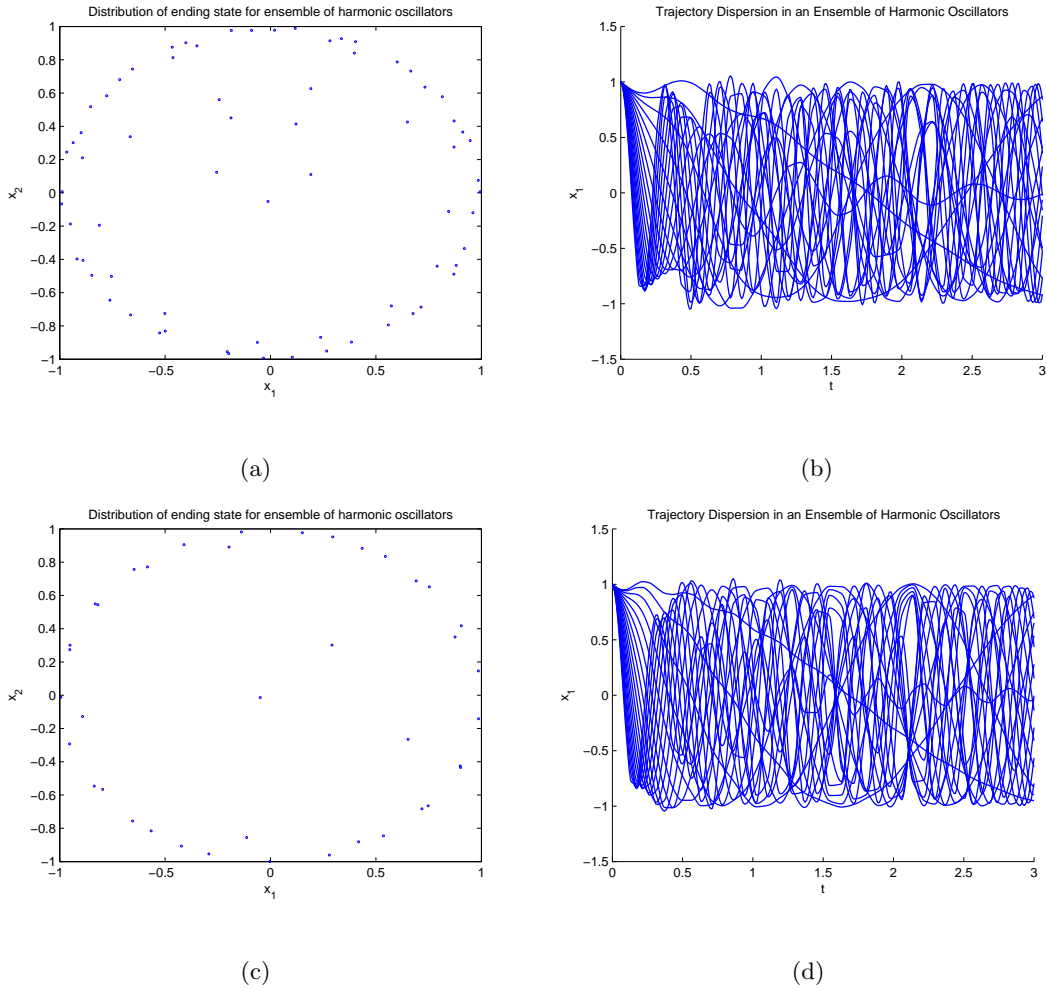
Figure 1.1: Dispersion of states for an ensemble of harmonic oscillators. Here a sample of state trajectories and end states is shown for a control designed using Lyapunov methods to stabilize a-b) the median case scenario and c-d) the worst case scenario.

on the value of the parameter for each individual system. Necessary and sufficient conditions for ensemble controllability are provided for a number of linear and bilinear systems in Ref. [47, 48], however such conditions for the general nonlinear ensemble are absent. Even when the ensemble can be shown to be controllable, determination of the desired control is difficult, and it is not surprising that such a control is available in closed

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form for only a small number of systems [48]. Therefore a new class of computational methods must be developed to provide numerical solutions to ensemble control problems.

One approach to the development of computational methods for ensemble control stems from the field of optimal control. This approach leverages existing direct methods for optimal control by formulating an objective functional for the ensemble control problem such that the solution to the resulting optimal control problem achieves the desired state transfer. One such framework is suggested in Ref. [48], wherein the goal is to minimize the expectation of the square error of the final state. Ruths [73,74,76] extends the pseudospectral optimal control method [29,69] to this problem by considering an approximation with a Legendre-Gauss-Lobatto (LGL) pseudospectral scheme in both the parameter and time domains, and provides analysis of the consistency properties of such a scheme.

By viewing the ensemble as a single system with stochastic parameters, the problem of ensemble control can be shown to be a member of the class of nonlinear optimal control problems with parameter uncertainty [73]. To demonstrate how this approach can be used to formulate an uncertain optimal control problem, consider the problem of trying to regulate the final state of the ensemble of harmonic oscillators with variation in the natural frequency. We can formulate this as an optimal control problem with parameter uncertainty by considering (1.7) as an uncertain dynamic system for a harmonic oscillator whose natural frequency is a random variable uniformly distributed on $[0, 20]$. We introduce a quadratic objective functional, which has the property that a minimizer to the objective will also reduce the amplitude oscillator while also keeping

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the expended control energy in reasonable bounds.

The uncertain optimal control problem is: find a state and control pair (x,u) which minimizes the objective functional

$$
\begin{aligned}
J &= \mathbb{E}^P \left[\beta \big[(x_1(T,\omega))^2 + (x_2(T,\omega))^2 \big] + \gamma \int_0^T \big[(u_1(t))^2 + (u_2(t))^2 \big] dt \right] \\
&= \beta \, \mathbb{E}^P \left[(x_1(T,\omega))^2 + (x_2(T,\omega))^2 \right] + \gamma \int_0^T \big[(u_1(t))^2 + (u_2(t))^2 \big] dt
\end{aligned}
$$

Subject to the uncertain dynamical system (1.7). Here β and γ are scale factors which weight the priority of minimizing the error of the final state against minimizing the expended control energy. This objective functional can be used to design a control which achieves an end state in a desired neighborhood of zero, therefore the problem of stabilizing this ensemble can be approached using the uncertain optimal control framework considered in this thesis.

1.3   Thesis Contributions

The uncertain optimal control Problem C formulated in this chapter has the potential to be applied to a wide variety of optimization problems, including trajectory optimization, optimal search theory, and ensemble control. Given the difficulty in solving standard nonlinear optimal control problems, it is not surprising that the inclusion of the uncertain parameter and expectation of the cost functional over the parameter space, combined with the nonlinear dynamics and control constraints, makes this problem particularly challenging. The literature presents several numerical methods for special cases of this problem which are based on direct methods for optimal control,

using either a Euler [13], Composite-Simpson [22, 23], or LGL-quadrature [73, 74, 76] discretization of the parameter space. However, given the range of possible applications which can be addressed in the uncertain optimal control setting, it is desirable to have a unified computational framework providing results for the entire class of uncertain optimal control problems.

The motivation of this dissertation is the development of this framework, including algorithms for the numerical solution of this class of problems as well as necessary conditions for validity and verification of numerically computed solutions. Our approach is based on discretization of the parameter space using a numerical scheme to approximate the expectation in the objective functional. The resulting approximate problem is a standard optimal control problem which can be solved using existing direct methods for standard optimal control problems. In this sense our framework is an extension of direct methods for computational optimal control to problems with parameter uncertainty. We expand upon previous approaches for the optimal search and ensemble control problems by demonstrating that any convergent quadrature scheme will produce a discretized problem which is a meaningful approximation of the original problem. Furthermore, we demonstrate a method based on sample average approximations which can be used to approach a problem with a large number of stochastic parameters. In addition, we determine necessary conditions for optimality for the class of uncertain optimal control problems, both in the form of an extension of Pontragin's minimum principle and an optimality function based on the $\mathcal{L}_2$-Frechet derivative of the objective functional. By extending a broad range of results from computational op-

timal control to the uncertain optimal control problem, we hope to provide a roadmap for new algorithms and approximation methods to be developed for the new problem setting with parameter uncertainty.

1.3.1 Numerical Methods for the Uncertain Optimal Control Problem

We provide a class of numerical methods for the solution of the uncertain optimal control problem based on a discretization scheme in the parameter space. In this approach, a set of nodes and weights are chosen from the parameter space, and the expectation in the objective functional is approximated by a finite sum. The result is a sequence of approximating nonlinear optimal control problems with a Bolza form objective functional. The advantage of this method is that the approximate problem can be solved using existing techniques for computational optimal control [29, 41, 79]. Although the application of this approximation scheme to the uncertain optimal control problem is straightforward, the numerical framework must be carefully analyzed as it is known that inappropriately designed discretization schemes for optimal control may produce incorrect results [14]. Therefore, each proposed discretization scheme must be demonstrated to provide an approximate control which is a reasonable approximation to the optimal control for the original problem. We provide this analysis for a variety of discretization schemes, focusing on classes of algorithms based on either numerical quadrature or Monte Carlo integration for the approximation of the objective functional. We contrast this work to consistency and convergence results on standard optimal control problems, for example results in Ref. [29, 41, 42, 61], as the discretization in this

work occurs in the parameter space rather than the time domain.

Some aspects of this analysis have been carried out in special cases for algorithms based on a numerical quadrature approximation of the uncertain optimal control problem. Ref. [22,23] uses a composite-Simpson integration scheme to discretize a two-dimensional parameter space and develops a computational method for solving a reduced version of this class of problems. They also analyze the performance of the computational method using Polak's consistent approximation theory [61]. Ref. [75] provides consistency and convergence results for a particular computational method based on a LGL-pseudospectral approximation in both the parameter and time domains. In this dissertation we extend these results by demonstrating that any convergent quadrature scheme will produce an approximation of the uncertain optimal control problem which is consistent in the sense that an accumulation point of a sequence of approximate global minimizers will be a global minimizer to the original problem. Establishing this property for a variety of quadrature schemes is important because the convergence properties of the state variables depend on the collocation nodes chosen for the parameter space [88].

While algorithms based on numerical quadrature may be efficient for problems with low-dimensional parameter spaces, such schemes are inherently limited by the curse of dimensionality. Indeed, the dimension of the approximate problem increases exponentially with an increase in the number of stochastic parameters, which renders solution of the approximate problem intractable for problems with even a modest number of parameters. This difficulty is inherent to the approximation of dynamical systems with stochastic parameters and other techniques such as polynomial chaos are also subject

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to the curse of dimensionality. Therefore, in many cases Monte Carlo simulation is required to approximate an uncertain dynamical system [6], and consistency results for a Monte Carlo based scheme for the uncertain optimal control problem are desirable.

We propose a sample average approximation approach to the uncertain optimal control problem which is applicable for high-dimensional problems. In this method, an independently distributed random draw is taken from the parameter space, and the expectation in the objective functional is approximated by the sample average. Early work on the sample average approximation approach to stochastic optimization can be found in [3,5]. These results provide a foundation for our analysis on the uncertainty optimal control problems. For a treatment of cases in finite dimensions; see [81]. When applied to a problem with a finite-dimensional decision space, sample average approximation produces a sequence of approximating nonlinear programming problems. When applied to the uncertain optimal control problem, it produces a sequence of nonlinear optimal control problems which can be solved using existing direct methods. Because the number of nodes sampled does not depend on the dimension of the parameter space, this method does not suffer from the same curse of dimensionality as the previously-considered quadrature method.

However, the difference in convergence properties between numerical quadrature and the strong law of large numbers for sample averages means that the analysis performed for the quadrature discretization scheme is not appropriate for the sample average scheme. Instead, we take an approach from Polak's seminal text on approximation of optimal control problems (see Polak [61, Chapter 4] or Section 2.3), which uses

tools and concepts from variational analysis. In this approach, the convergence proper-

ties of the approximate problem are established by demonstrating the epiconvergence of

the sequence of approximate objective functionals. We provide a consistency result for

the uncertain optimal control framework using an extension of the strong law of large

numbers to random lower semicontinuous functions.

1.3.2   Necessary Conditions for the Uncertain Optimal Control Problem

Many algorithms for the numerical solution of the standard optimal control are

based on necessary conditions for optimality such as the Pontryagin minimum principle.

Research into such necessary conditions may also provide insight into the nature of the

solutions of the uncertain optimal control problem as well as new numerical tools. In ad-

dition, necessary conditions can be used for validation and verification of a numerically

computed solution. In this work we provide necessary conditions for the uncertain op-

timal control problem by extending existing theoretical results on the standard optimal

control problem with Bolza cost. We provide two separate sets of necessary conditions.

The first is a Pontraygin-like necessary condition derived by analyzing the convergence

properties of the dual problem for an approximation based on numerical quadrature.

The second is an optimality function based on the $\mathcal{L}_2$-Frechet derivative of the objective

functional for the original uncertain optimal control problem.

For the uncertain optimal control Problem C discussed in Section 1.3.1 a

Pontryagin-like necessary condition is provided using the quadrature numerical method.

In this approach the parameter space is discretized and the expectation in the objective functional is approximated using a quadrature scheme, resulting in a sequence of approximating standard optimal control problems. Because these problems have a Bolza-type objective functional, they can be dualized using the Hamiltonian and adjoint variables of Pontryagin's minimum principle. We demonstrate convergence properties of this sequence of dual problems by analyzing the convergence of the approximate Hamiltonians and adjoint variables. This analysis provides a Pontryagin-like Hamiltonian minimization property for optimal solutions for the original problem which are accumulation points of a sequence optimal solutions to the approximation problem. This condition can be used for verification of numerical solutions for the uncertain problem which are calculated using the proposed quadrature framework.

In addition, we use the approach of Polak [61] to develop a necessary condition based on the $\mathcal{L}_2$-Frechet derivative of the objective functional, as well related convergence results. The necessary condition is given in the form of an optimality function, which is a nonpositive upper semicontinuous function which must be zero when evaluated at any optimal solution. A decision variable which satisfies this necessary condition is called stationary. We demonstrate that the sample average approximation scheme will produce a sequence of approximate optimality functions which are epiconvergent, which implies that an accumulation point of a sequence of approximately stationary points is stationary. We therefore demonstrate that the scheme based on sample average approximations is consistent in the sense of Polak [61, Section 3.3].

1.4   Outline

The dissertation is organized as follows. Chapter 2 introduces preliminary mathematical concepts and tools which are used to establish the main results of the dissertation. Chapter 3 introduces an uncertain optimal control problem in which the uncertainty occurs only in the objective functional as well a quadrature based numerical method and necessary conditions for this problem. The results are also extended to problems with uncertain parameters in the dynamics and initial conditions in Section 3.6 of this Chapter. Chapter 4 provides a numerical method for the uncertain optimal control problem based on sample averages, as well as a necessary condition based on the $\mathcal{L}_2$-Frechet derivative of the objective functional. Chapter 5 provides concluding remarks for the dissertation as well as comments on future research.

Chapter 2

Preliminary Mathematical Concepts and Tools

In this chapter we review concepts and mathematical tools which are essential in establishing the main results of this work. The focus of this dissertation is the development of a computational framework for the numerical solution of optimal control problems with stochastic parameters, therefore in this chapter we hope to familiarize the reader with recent work in the fields of computational optimal control and stochastic optimization.

Section 2.1 introduces Pontryagin's minimum principle, a set of necessary conditions for nonlinear optimal control problems which can be used to verify and validate numerically computed solutions, or alternatively to demonstrate the validity of a numerical algorithm for solving optimal control problems. Section 2.2 introduces the class of direct computational optimal control methods, wherein the continuous-time control

problem is approximated by a discretized high-dimensional constrained nonlinear optimization problem. Section 2.3 discusses Polak's theory of consistent approximations for optimal control problems [61], which uses tools from variational analysis to demonstrate the validity of a class of direct methods for optimal control. Section 2.4 introduces sample average approximations, a method which can be used to approximate optimization problems with stochastic parameters using Monte Carlo integration techniques.

2.1 Optimal Control and Pontryagin's Minimum Principle

In this section we introduce the standard nonlinear optimal control problem as well as Pontryagin's Minimum Principle, a theoretical result which is the underlying basis of a number of algorithms for the solution of this problem. The goal in the standard nonlinear optimal control problem is to find, among all admissible state-and-control pairs for a dynamical systems, the pair which will achieve the minimum of a predetermined cost functional. A state-and-control pair is said to be admissable if it satisfies the dynamical system as well as a set of pointwise inequality constraints on the control. In this work we consider systems in which both the dynamical system and inequality contraints may be nonlinear. A standard optimal control problem can be stated as follows:

Find a state-control pair (x, u) to minimize the objective functional

$$J(x, u) = F(x(T)) + \int_0^T r(x(t), u(t))dt, \tag{2.1}$$

subject to the dynamical system

$$\dot{x}(t) = f(x(t), u(t)), \qquad\qquad x(0) = x_0, \qquad\qquad (2.2)$$

and the control constraint

$$g(u(t)) \leq 0 \text{ for all } t \in [0, T]. \qquad\qquad (2.3)$$

Here $x : [0, T] \mapsto \mathbb{R}^{n_x}, u : [0, T] \mapsto \mathbb{R}^{n_u}, F : \mathbb{R}^{n_x} \to \mathbb{R}, r : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$, and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_x}$.

The form of the objective functional (2.1), which consists of both an endpoint cost and a running cost, is referred to as a Bolza type cost functional. Note that it is possible for an optimal control problem to have state dependent constraints, for example, pure state constraint $p(x(t)) \leq 0$ for all $t \in [0, T]$, or mixed state-control constraints, $h(x(t), u(t)) \leq 0$ for all $t \in [0, T]$. In this thesis, we limit our discussion on control constraint only.

For the considered constrained optimal control problem, the celebrated Pontryagin's Minimum Principle provides a set of necessary condition for the optimal solution. Under mild regularity conditions, Pontryagin [63] proved that every local optimal solution $(x^*(t), u^*(t))$ associates to a costate (dual) variable, $\lambda(t)$. The primal optimal solution, $(x^*(t), u^*(t))$, and the dual variable, $\lambda(t)$, satisfy a differential-algebraic equation with certain boundary conditions called transversality conditions. For the considered control-constrained optimal control problem, the necessary conditions are summarized in the following.

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Proposition 2.1.1. *Pontryagin's Minimum Principle [63]*

Let (x^*, u^*) be an optimal solution to the problem defined by (2.1-2.3). Then there exists an absolutely continuous costate variable $\lambda : [0,1] \mapsto \mathbb{R}^x$ and Hamitonian $\mathbf{H} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \mapsto \mathbb{R}$ such that the following are satisfied:

$$\mathbf{H}(x, \lambda, u) = f(x, u)^T \lambda + r(x(t), u(t)),$$

$$\dot{x} = \frac{\partial \mathbf{H}}{\partial \lambda}, \qquad x(0) = x_0,$$

$$\dot{\lambda} = -\frac{\partial \mathbf{H}}{\partial x}, \qquad \lambda(1) = F_x(x(T)).$$

Furthermore u^* satisfies the Hamiltonian minimization condition

$$u^*(t) \in \arg \min_{g(u) \leq 0} \mathbf{H}(x^*, \lambda^*, u) \qquad \text{for almost every } t \in [0, T].$$

For some simple optimal control problems, for example, linear quadratic problems, the state-adjoint system for x and λ lends itself to a straightforward solution when the Hamiltonian minimization criterion is applied. In these cases the necessary condition provides a tool to determine a closed-form solution to the problem. However, for most nonlinear or constrained optimal control problems, the closed form solution to the necessary conditions cannot be analytically obtained; and numerical algorithms are needed to compute the optimal solutions.

2.2 Computational Optimal Control

It is well-known that an optimal control problem can be very difficult to solve analytically when the dynamics associated with the system to be controlled are nonlinear

[11,12]. When the boundary value problem provided by Pontryagin's minimum principle does not admit a closed form solution, it may be possible to compute a numerical solution using an existing discretization technique for the solution of ordinary differential equation. The resulting approximate solution to the dual problem (necessary conditions) will converge as the number of nodes used in the discretization increases. This process of dualization and discretization of the optimal control problem is referred to as an *indirect method* for the numerical solution an optimal control problem and has been used successfully to solve control problems from a wide range of application areas. However, indirect methods for optimal control pose several computational difficulties. First, the dualization step in this process may be very labor intensive and must be performed by an individual knowledgeable in the field of optimal control. Second, the boundary value problem resulting from the dualization step is extremely sensitive to the initial guess for the optimal control.

In the last decades, great progress has been made in the development of a class computational algorithms for constrained nonlinear optimal control problems which are based on the direct discretization of the time domain. These so-called *direct methods* avoid the numerical instability inherent in indirect schemes, and in addition do not require the user to explicitly determine the state-adjoint equations for the control problem. Instead, the direct scheme uses quadrature to approximate the objective functional and collocation to approximate the dynamics for the new discretized time variable. The result is a high-dimensional nonlinear programming problem which can be solved using existing techniques. A variety of computational algorithms have been developed based

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on direct methods, using discretization schemes including Euler [61], Chapter 4, Runge-Kutta [41, 79], and Pseudospectral [29, 42, 69]. These computational optimal control methods have achieved great success in many areas of control applications [7, 13, 40, 54].

To demonstrate such a direct discretization method, consider the Euler scheme, in which the problem is discretized by selecting N number of nodes in the time domain. For simplicity, the nodes are assumed to be uniformly distributed; and the dimensions of the state and control are assumed to be 1, i.e., $n_x = n_u = 1$. The state trajectory $x(t)$ is approximated by the vector $x^N = (x_1^N, x_2^N, \ldots x_N^N) \in \mathbb{R}^N$ and the control $u(t)$ is approximated by the vector $u^N = (u_1^N, \ldots, u_N^N) \in \mathbb{R}^N$. Here (x_i^N, u_i^N) are approximated state and control at node t_i, i.e., $(x(t_i), u(t_i)) \approx (x_i^N, u_i^N)$.

The approximate optimal control problem is then: find $(x^N, u^N) \in \mathbb{R}^N \times \mathbb{R}^N$ which minimizes the objective functional

$$J^N(x^N, u^N) = F(x_N^N) + \sum_{i=1}^{N} r(x_i^N, u_i^N)\Delta t, \tag{2.4}$$

subject to the difference equation

$$x_{i+1}^N = x_i^N + f(x_i^N, u_i^N)\Delta t, \qquad x_1^N = x_0, \qquad \Delta t = \frac{T}{N}, \tag{2.5}$$

and the control constraint $g(u_i^N) \leq 0$ for every $i \in \{1, \ldots, n\}$.

The advantage of this approach is that the approximate problem is a nonlinear programming problem and can therefore be solved using existing methods such as sequential quadratic programming. However, the apparent simplicity of such an approximation scheme belies deep theoretical issues inherent in the approximation of

optimal control problems. Indeed, it can be demonstrated that for nonlinear optimal control problems, an inappropriately designed discretization scheme may not be convergent [14]. Therefore particular care must be taken when determining a numerical scheme for solution of nonlinear optimal control problems, and any chosen scheme must be proven to provide an appropriate approximation to the problem. For example, Hager [31] demonstrates that a standard Runge-Kutta discretization for the numerical solution of dynamical systems may fail to provide the correct solution when used to discretize an optimal control problem. By employing Pontryagin's minimum principle to analyze the convergence of the adjoint variables, he is able to demonstrate that adding additional constraints to the coefficients used in the discretization scheme will guarantee the convergence of the solution of the approximate problem to the optimal control.

2.3   Consistent Approximation of Optimal Control Problems

In this dissertation we draw heavily on results from Polak's seminal text on the consistent approximation of optimal control problems [61]. In this work, Polak presents a theoretical framework to assess the convergence properties of discretization schemes for optimal control problems using concepts from the field of variational analysis. In the absence of convexity, it is generally not possible to determine whether a discretization scheme will lead to a numerical solution which is a meaningful approximation of the optimal control for the original problem. Polak introduces the concept

of consistency, a property which guarantees that an accumulation point of a sequence of approximately optimal solutions will be an optimal solution to the original problem. To establish the consistency result for an approximation scheme he uses the notion of epiconvergence, which is a natural setting to address the approximation of an objective functional because an epiconvergent sequence preserves some properties of the inf and arg min operators. For the purpose of this section, we will refer to the admissable set of our optimization problem as $\mathbf{U}$, the decision variables as $\eta \in \mathbf{U}$, and the objective functional as $h : \mathbf{U} \mapsto \mathbb{R}$.

Definition 1. *[4] Let $(\mathbf{U}, d)$ be a separable complete metric space. Consider the sequence of lower semi-continuous functions $h_M : \mathbf{U} \mapsto \mathbb{R}$. We say that h_M epiconverges to h, denoted $h_M \to^{epi} h$, if and only if*

i) $\liminf h_M(\eta_M) \geq h(\eta)$ *whenever* $\eta_M \to \eta$,

ii) $\lim h_M(\eta_M) = h(\eta)$ *for at least one sequence* $\eta_M \to \eta$

The following proposition demonstrates that a numerical scheme which provides an epiconvergent sequence of approximate objective functionals is appropriate to solve a nonlinear optimization problem.

Proposition 2.3.1. *[4, Theorem 2.5] Theorem 2.5. Let $(\mathbf{U}, d)$ be a separable complete metric space. Consider the sequence of lower semi-continuous functions $h_M : \mathbf{U} \mapsto \mathbb{R}$. Suppose that h_M epiconverges to h. If $\{\eta^M\}_{M \in \mathbb{N}} \subset \mathbf{U}$ is a sequence of global minimizers to h_M, and $\hat{\eta}$ is any accumulation point of this sequence (along a subsequence indexed*

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by a set $K \subset \mathbb{N}$), then $\hat{\eta}$ is a global minimizer of h and $\lim_{M \in K} \inf_{\eta \in \mathbf{U}} h_M(\eta^M) = \inf_{\eta \in \mathbf{U}} h(\eta)$.

Proposition 2.3.1 can be used to show that the Euler scheme defined by (2.4)-(2.5) can be used to determine the optimal solution of the optimal control problem (2.1)-(2.3). By evaluating the epiconvergence of the approximate objective functional (2.4) it can be shown an appropriate method of solution of the original optimal control problem is to solve the approximate problem with N nodes and analyze the convergence properties of the sequence $\{u_N^*\}_{N=1}^{\infty}$ of approximate optimal controls. If a sequence of approximately optimal controls $\{u_N^*\}_{N=1}^{\infty}$ converges to a control u^{∞}, then u^{∞} is known to be an optimal solution of the original optimal control problem.

An additional component of Polak's theory of consistent approximations is the development of a necessary condition and analysis of its convergence properties as the number N of nodes used in the approximation approaches infinity. He presents an alternative necessary condition to Pontryagin's minimum principle based on optimality functions derived using the $\mathcal{L}_2$-Frechet derivative of the objective functional.

Definition 2. *[61] Consider the problem of finding $\eta \in \mathbf{U}$ to minimize the objective functional $h : \mathbf{U} \mapsto \mathbb{R}$. An upper semi-continuous function $\theta : \mathbf{U} \mapsto \mathbb{R}$ is an optimality function for this problem if:*

i) $\theta(\eta) \leq 0$ for all $\eta \in \mathbf{U}$.

ii) If η is a local minimizer of B, then $\theta(\eta) = 0$.

A point that satisfies $\theta(\eta) = 0$ is called a *stationary point* of the optimality function, and it is clear from this definition that every optimal solution to this problem must be stationary. An approximation is called consistent if it approximates both the objective functional and optimality function well.

Definition 3. *[61] Let* $\mathbf{U}$ *be a complete separable metric space, let* $h^M : \mathbf{U} \mapsto \mathbb{R}, h :$ $X \mapsto \mathbb{R}$ *be lower semi-continuous functions, and let* $\theta^M : \mathbf{U} \mapsto \mathbb{R}, \theta : \mathbf{U} \mapsto \mathbb{R}$ *be non-positive upper semi-continuous functions. We say that the pair* $\{h^M, \theta^M\}_{M \in \mathbb{N}}$ *is a consistent approximation to the pair* $\{h, \theta\}$ *if:*

i) $h_M \rightarrow^{epi} h.$

ii) *If* $\{\eta_M\}_{M=1}^{\infty}$ *is a sequence converging to* η, *then* $\limsup_{M \to \infty} \theta_M(\eta_M) \leq \theta(\eta)$.

Note that the convergence property in Definition 6.ii guarantees that in a consistent approximation, an accumulation point of a sequence of approximately stationary points will be a stationary point of the original problem.

2.4   Sample Average Approximations

Sample average approximation is a technique for the solution of an optimization problem in which the goal is to minimize the expectation of a predetermined cost functional over a space of stochastic parameters. Consider the problem of finding a decision variable $\eta \in \mathbf{U}$ which minimizes the objective functional $\mathbb{E}^P[h(\eta, \omega)]$, where (Ω, P, Σ) is a probability space and $h : \mathbf{U} \times \Omega \mapsto \mathbb{R}$. The sample average approach is to

take an independent P-distributed draw $\{\omega_i^M\}_{i=1}^M$ from Ω and approximate the objective functional $\mathbb{E}^P[h(\cdot, \omega)]$ by the sample average $\frac{1}{M} \sum_{i=1}^M h(\cdot, \omega_i^M)$. For a treatment of sample average approximation techniques for problems with finite dimensional decision spaces, see Ref. [81].

While the strong law of large numbers guarantees the almost sure convergence of the approximate objective values for a fixed decision variable, this does not guarantee convergence of the sequence of approximate problems. Ref. [3,5] provide an extension of the strong law of large numbers to random lower semincontiuous functions which can be used to determine the convergence properties of the sequence of approximate objective functionals. This allows the consistency properties of the approximation scheme to be analyzed using Polak's theory of consistent approximations [61], discussed in Section 2.3.

Definition 4. *[5] Let $(\mathbf{U}, d)$ be a separable complete metric space with $\mathcal{B}$ the Borel field generated by the open subsets of $\mathbb{U}$. Let P be a probability measure on the measurable space (Ω, Σ) such that Σ is P-complete. A function $h : \mathbf{U} \times \Omega \mapsto \mathbb{R}$ is a random lower semi-continuous if and only if:*

i) for all $\omega \in \Omega$, the function $\eta \mapsto h(\eta, \omega)$ is lower semi-continuous,

ii) $(\eta, \omega) \mapsto h(\eta, \omega)$ is $\mathcal{B} \otimes \Sigma$ measurable.

In probability theory, the strong law of large numbers guarantees the almost sure convergence of the sample average as the number of samples drawn approaches infinity. The following proposition extends this result to random lower semi-continuous

functions.

Proposition 2.4.1. *[3, Theorem 2.3] Let (Ω, Σ, P) be a probability space such that Σ is P-complete. Let $(\mathbf{U}, d)$ be a separable complete metric space. Suppose that the function $h : \mathbf{U} \times \Omega \mapsto \mathbb{R}$ is a random lower semi-continuous function and there exists an integrable function $a_0 : \Omega \mapsto \mathbb{R}$ such that $h(\eta, \omega) \geq a_0(\omega)$ almost surely. Let $\{\omega_1, \ldots, \omega_M\}$ be an independent P-distributed random draw and define*

$$\hat{h}(\eta, \omega_1, \ldots, \omega_M) = \frac{1}{M} \sum_{i=1}^{M} h(\eta, \omega_i).$$

Then, as $M \to \infty, \hat{h}(x, \omega_1, \ldots, \omega_M)$ epiconverges almost surely to $\mathbb{E}^P[h(\eta, \omega)]$.

This result, combined with the theory of consistent approximations introduced in Section 2.3, is used evaluate the validity of a sample average scheme for optimal control by demonstrating the epiconvergence of the objective functionals and optimality functions in Chapter 4.

Chapter 3

Quadrature Approximation of an

Uncertain Optimal Control Problem

In this chapter we introduce a class of optimal control problems which incorporate parameter uncertainty into the cost in the form of an expectation over a space of stochastic parameters in the objective functional. These problems may be encountered in applications from optimal search for moving targets [22, 23, 51, 52]. We consider a class of algorithms for the numerical solution of these problems based on a quadrature approximation of the expectation in the objective functional. In this approach, a set of nodes and weights are chosen from the parameter space, and the expectation is approximated by a finite sum. The advantage of this method is that the resulting approximate problem is a standard optimal control problem which can be solved using existing techniques [29, 41, 79]. Our goal is to provide a rigorous analysis of the convergence properties of these algorithms as well as necessary conditions which must be

satisfied by the solutions.

Some aspects of this analysis have been carried out in special cases. Ref. [22, 23] uses a composite-Simpson integration scheme to discretize a two-dimensional parameter space and develops a computational method for solving a reduced version of this class of problems. They also analyze the performance of the computational method using Polak's consistent approximation theory [61], Section 3.3. In this chapter we extend these results by demonstrating that any convergent quadrature scheme will produce an approximation of the uncertain optimal control problem which is consistent in the sense of Polak [61]. Establishing this property for a variety of quadrature schemes is important because the convergence properties of the state variables depend on the collocation nodes chosen for the parameter space [6]. In addition, we provide a Pontryagin-like necessary condition which must be satisfied by an optimal solution computed by the given numerical method. We contrast this work to consistency and convergence results on standard optimal control problems, for example results in Ref. [29, 41, 42, 61], as the discretization in this work occurs in the parameter space rather than the time domain.

Although we focus in this chapter on a class of problems in which the uncertainty occurs only in the objective functional, this approach can be extended to problems with uncertainty in the agent dynamics and initial state. An analysis of such an extension is given in Section 3.6.

3.1 Formulation of the Uncertain Optimal Control Problem

In a standard nonlinear optimal control problem, the objective functional is of the Bolza type, which consists of an end cost as well as an integral over the time domain. In this section we introduce a class of non-standard optimal control problems in which the objective functional involves an expectation of a Bolza-type integral over a space of stochastic parameters:

Problem B. Determine the function pair $\{x, u\}$ with $x \in W_{1,\infty}([0,1]; \mathbb{R}^{n_x}), u \in L_{\infty}([0,1]; \mathbb{R}^{n_u})$ that minimizes the cost functional

$$J = \int_{\Omega} \left[F\left(x(1), \omega\right) + G\left(\int_0^1 r(x(t), u(t), t, \omega) dt \right) \right] p(\omega) d\omega$$

subject to the dynamics

$$\dot{x}(t) = f(x(t), u(t)), \qquad (3.1)$$

initial condition $x(0) = x_0$, and the control constraint $g(u(t)) \leq 0$ for all $t \in [0,1]$.

In Problem B, $W_{1,\infty}([0,1]; \mathbb{R}^{n_x})$ is the space of all essentially bounded functions with essentially bounded distributional derivatives, which map the interval $[0,1]$ into the space $\mathbb{R}^{n_x}$, and $L_{\infty}([0,1]; \mathbb{R}^{n_u})$ is the set of all essentially bounded functions. The function p is a continuous probability density function for the stochastic parameter $\omega \in \Omega \subset \mathbb{R}^{n_\omega}$ and we allow r to be vector valued: that is, $r : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^1 \times \mathbb{R}^{n_\omega} \mapsto \mathbb{R}^K, G : \mathbb{R}^K \mapsto \mathbb{R}$.

Given the difficulty in solving standard nonlinear optimal control problems,

it is not surprising that the inclusion of the expectation of the cost functional over the parameter space, combined with the nonlinear dynamics and control constraints, makes Problem B particularly challenging. In this work we propose a computational framework for the solution of the uncertain optimal control Problem B. Based on the numerical approximation of the integral over the stochastic parameters in the objective functional, the considered uncertain optimal control problem can be approximated by a sequence of standard nonlinear optimal control problems, which can in turn be solved using existing computational methods such as Runge-Kutta [41, 79] and pseudospectral [29] approaches. To ensure meaningful results in this computational framework, it is essential to guarantee that the discretization schemes provide valid approximations to the original non-standard optimal control Problem B. Indeed, even for standard optimal control problems, there are counterexamples showing that an inappropriately designed discretization may not be convergent [14]. In this chapter, we show that the proposed computational framework approximates the optimal solution to the non-standard optimal control problem under mild assumptions. In particular, we show in Section 3.3 that the approximation based on the discretization process satisfies a zeroth-order consistency property. That is, accumulation points of a sequence of optimal solutions to the approximate problem are optimal solutions to the original uncertain optimal control problem.

3.2 Discretization of the Uncertain Optimal Control Problem

In this section we present a computational framework for solving the non-standard optimal control Problem B by using a numerical scheme to approximate the integral over the stochastic parameters in the objective functional. The following regularity conditions are assumed.

Assumption 1. *The function $g : \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_g}$ used in the definition of the control constraint is continuous and the set $U = \{\nu \in \mathbb{R}^{n_u} | g(\nu) \leq 0\}$ is compact.*

In a real world scenario the set of allowable controls will be bounded and therefore U, being a closed and bounded set, will be compact.

Assumption 2. *Let $\mathcal{A}$ be the set of feasible pairs to Problem B, that is the set of all $\{x, u\}$ with $x \in W_{1,\infty}([0,1]; \mathbb{R}^{n_x})$, $u \in L_\infty([0,1]; \mathbb{R}^{n_u})$ such that $u(t) \in U$ and $x(t) = x_0 + \int_0^t f(x(s), u(s)) ds$ for all $t \in [0,1]$. Then there exists a compact set $X \subset \mathbb{R}^{n_x}$ such that for each feasible pair $\{x, u\} \in \mathcal{A}$ we have $x(t) \in X$ for all $t \in [0,1]$.*

This assumption essentially requires for all bounded controls that there is no finite escape time. A large class of nonlinear systems satisfy this assumption, for example, input-to-state stable systems and systems for which f is globally Lipschitz or satisfies a linear growth condition.

Assumption 3. *The functions f, r and G are C^1. The set Ω is compact. Moreover, for the compact sets X and U defined in Assumptions 1-2 and for each $t \in [0,1]$, $\omega \in \Omega$,*

the Jacobian $r_x(\cdot, \cdot, t, \omega)$ is Lipschitz on the set $X \times U$, and the corresponding Lipschitz constant is uniformly bounded in ω and t. The function $F(\cdot, \omega)$ is C^1 on X for all $\omega \in \Omega$; in addition, F and $\nabla_x F$ are continuous with respect to ω.

To approximate the integral over the stochastic parameters in the objective functional in Problem B, we introduce numerical integration schemes that satisfy the following assumption.

Assumption 4. *For each $M \in \mathbb{N}$, there is a set of nodes $\{\omega_i^M\}_{i=1}^M \subset \Omega$ and an associated set of weights $\{\alpha_i^M\}_{i=1}^M \subset \mathbb{R}$, such that for any continuous function $h : \Omega \to \mathbb{R}$,*

$$\int_\Omega h(\omega)d\omega = \lim_{M \to \infty} \sum_{i=1}^M h(\omega_i^M)\alpha_i^M.$$

Throughout this chapter, M is used to denote the number of nodes used in the numerical integration scheme. Many numerical integration schemes, e.g., numerical quadrature and Simpson's rule, satisfy Assumption 8 and are applicable to determine the nodes and weights.

Remark 1. *Note that if $h_M : \Omega \to \mathbb{R}$ is continuous for all $M \in \mathbb{N}$ and $\{h_M\}$ converges uniformly to h, then*

$$\int_\Omega h(\omega)d\omega = \lim_{M \to \infty} \sum_{i=1}^M h_M(\omega_i^M)\alpha_i^M.$$

This property is frequently used later.

Once the numerical scheme is chosen, the integral over the parameter space is approximated by a sum; and an approximate objective functional for each $M \in \mathbb{N}$ can

be defined by

$$J^M = \sum_{i=1}^{M} \left[F\left(x(1), \omega_i^M\right) + G\left(\int_0^1 r(x, u, t, \omega_i^M) dt \right) \right] p(\omega_i^M) \alpha_i^M. \qquad (3.2)$$

Now we are ready to define the approximate optimal control problem:

Problem B$^{\mathbf{M}}$.   Determine the function pair $\{x, u\}$, where $x \in W_{1,\infty}([0, 1]; \mathbb{R}^{n_x})$, and $u \in L_\infty([0, 1]; \mathbb{R}^{n_u})$, that minimizes the cost functional (3.2) subject to the dynamics (3.1) and the control constraint $g(u(t)) \le 0$ for all $t \in [0, 1]$.

We now show that Problem B^M can be reformulated as a standard optimal control problem with a Bolza form objective functional. To this end we introduce the auxiliary variable $z : [0, 1] \times \Omega \mapsto \mathbb{R}$ governed by the dynamics

$$\dot{z}(t, \omega) = r(x(t), u(t), t, \omega), \ \ z(0, \omega) = 0, \ \forall \ \omega \in \Omega. \qquad (3.3)$$

So that $z(1, \omega) = \int_0^1 r(x(t), u(t), t, \omega) dt$. By forming the vector

$$\zeta_M(t) = [z(t, \omega_1^M), \dots, z(t, \omega_M^M)]^T,$$

we can reformulate the objective functional (3.2) as:

$$\hat{J}^M = \sum_{i=1}^{M} \left[F\left(x(1), \omega_i^M\right) + G(\zeta_{M,i}(1)) \right] p(\omega_i^M) \alpha_i^M. \qquad (3.4)$$

This is a Bolza objective functional with an end cost. Therefore Problem B^M is equivalent to the standard optimal control problem of finding a triplet $\{x, \zeta_M, u\}$ which minimizes the objective functional (3.4) subject to the dynamics

$$\dot{x}(t) = f(x(t), u(t))$$

$$\dot{\zeta}_{M,i}(t) = r(x(t), u(t), t, \omega_i^M), \ \ i = 1, \dots M$$

initial condition $x(0) = x_0$, $\zeta_{M,i}(0) = 0$, and the control constraint $g(u(t)) \leq 0$ for all $t \in [0,1]$. This formulation is used again when deriving a necessary condition for Problem B.

By using a numerical scheme to approximate the integral in the objective functional, the non-standard optimal control Problem B is discretized into a sequence of standard optimal control problems, Problem B^M. Problem B^M can be solved by existing computational optimal control methods, such as Runge-Kutta [41, 79], pseudospectral [29] methods, and indirect [8, 12] type of methods.

3.3 Convergence Properties of The Discretized Problem

It is well known in computational optimal control that a convergent numerical scheme for solving ODEs may be divergent when applied to optimal control problems [14, 29, 61]. Similarly, the convergence of the numerical integration assumed in Assumption 8 does not necessarily imply solutions of Problem B^M converge to solutions of the original Problem B. The focus of this section is to show that, under Assumptions 1 – 4, accumulation points of a sequence of optimal solutions to the approximate Problem B^M as the number of nodes M tends to infinity, are optimal solutions to Problem B. This consistency property guarantees that Problem B^M is indeed an appropriate approximation to Problem B.

Before introducing the main convergence result, we first make a note on the notation to be used. We define the set $\mathcal{N}_\infty^\# = \{V \subset \mathbb{N} | V \text{ infinite}\}$. That is, $\mathcal{N}_\infty^\#$ is the set

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of all subsequences of $\mathbb{N}$ of infinite length, which are designated by the index set $V \in \mathbb{N}$. When $M \to \infty$ as usual in $\mathbb{N}$, we write $\lim_{M\to\infty}$. However, in the case of convergence with respect to a subsequence designated by an index set V, we write $\lim_{M\in V}$. For sequences of feasible pairs $\{x_M, u_M\}$, the notation $\lim_{M\to\infty}\{x_M, u_M\} = \{x, u\}$ will mean that $\{x_M, u_M\}$ converges pointwise to $\{x, u\}$. Similarly $\lim_{M\in V}\{x_M, u_M\} = \{x, u\}$ will refer to pointwise convergence of the state-control pair along the subsequence indexed by V.

Lemma 3.3.1. *Suppose that Assumptions 1-3 hold. Then $\mathcal{A}$, the set of feasible pairs to Problem B defined in Assumption 2, is closed in the topology of pointwise convergence.*

Proof. Suppose that a sequence $\{x_M, u_M\} \subset \mathcal{A}$ and $\lim_{M\to\infty}\{x_M, u_M\} = \{x, u\}$. By the continuity of g, $u(t) \in U$ for all $t \in [0, 1]$. Note that because f is C^1, it is Lipschitz continuous on the compact set $X \times U$. Now consider

$$\left\| x(t) - x_0 - \int_0^t f(x(s), u(s))ds \right\| = \lim_{M\to\infty} \left\| x(t) - \int_0^t f(x(s), u(s))ds - x_M(t) \right.$$
$$\left. + \int_0^t f(x_M(s), u_M(s))ds \right\|$$
$$\leq \lim_{M\to\infty} L \int_0^t \|x(s) - x_M(s)\| + \|u(s) - u_M(s)\| \, ds$$
$$+ \|x(t) - x_M(t)\| ,$$

where L is the Lipschitz constant of f. Because $x(s), x_M(s) \in X$ and $u(s), u_M(s) \in U$, where X and U are compact, $\|x(s) - x_M(s)\|$ and $\|u(s) - u_M(s)\|$ are bounded for all $s \in [0, 1]$ and $M \in \mathbb{N}$. Therefore by the dominated convergence theorem,

$$x(t) = x_0 + \int_0^t f(x(s), u(s))ds$$

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for all $t \in [0, 1]$. Hence, $\{x, u\} \in \mathcal{A}$.   □

Lemma 3.3.1 shows that if $\{x_M, u_M\}$ is a sequence of feasible pairs for Problem B, then any accumulation point of this sequence is a feasible pair. It sets the foundation for the following result.

Theorem 3.3.2. *Suppose that Assumptions 1-8 hold. In addition, suppose that there exists $V \in \mathcal{N}_\infty^\#$ and a set of optimal pairs $\{x_M^*, u_M^*\}_{M \in V}$ for Problem B^M such that*

$$\lim_{M \in V} \{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}.$$

Then $\{x^\infty, u^\infty\}$ is an optimal solution to Problem B.

Proof. By Lemma 3.3.1, $\{x^\infty, u^\infty\}$ is a feasible solution to Problem B. Next, we prove the optimality of $\{x^\infty, u^\infty\}$. From Assumption 3, r is bounded and Lipschitz on $X \times U \times [0, 1] \times \Omega$ and G is uniformly continuous on $r(X, U, [0, 1], \Omega)$. From the Lipschitz continuity of r, we have, for all $\omega \in \Omega$

$$\int_0^1 \|r(x_M^*(t), u_M^*(t), t, \omega) - r(x^\infty(t), u^\infty(t), t, \omega)\| dt$$

$$\leq L \int_0^1 \|x_M^*(t) - x^\infty(t)\| + \|u_M^*(t) - u^\infty(t)\| dt.$$

By the dominated convergence theorem,

$$\lim_{M \in V} \int_0^1 \|x_M^*(t) - x^\infty(t)\| + \|u_M^*(t) - u^\infty(t)\| \, dt = 0$$

and this convergence must be uniform in ω. Then by the uniform continuity of G and the continuity of F, for each $\epsilon > 0$, there must exist $N \in \mathbb{N}$ such that for each $M \in V$

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with $M > N$ the following statements hold for all $\omega \in \Omega$

$$\left\| G\Big(\int_0^1 r(x_M^*(t), u_M^*(t), t, \omega)dt \Big) - G\Big(\int_0^1 r(x^\infty(t), u^\infty(t), t, \omega)dt \Big) \right\| < \frac{\epsilon}{2},$$

$$\| F(x_M^*(1), \omega) - F(x^\infty(1), \omega) \| < \frac{\epsilon}{2}.$$

This implies, by the statement in Remark 1,

$$\lim_{M \in V} J^M(x_M^*, u_M^*) = J(x^\infty, u^\infty).$$

Let $\{x, u\}$ be an arbitrary feasible pair for Problem B. Then, based on the optimality

of $\{x_M^*, u_M^*\}$, $J^M(x_M^*, u_M^*) \le J^M(x, u)$ for all $M \in V$. Thus

$$J(x^\infty, u^\infty) = \lim_{M \in V} J^M(x_M^*, u_M^*) \le \lim_{M \in V} J^M(x, u) = J(x, u).$$

Therefore $\{x^\infty, u^\infty\}$ is an optimal pair for Problem B, since it produces the minimum

cost among all feasible solutions. $\square$

Theorem 3.3.2 shows that if a subsequence of optimal solutions to Problem

B^M converges, this limit point is an optimal solution to Problem B. Based on Theorem

3.3.2, one can apply existing computational optimal control algorithms to solve Problem

B^M. If the solution sequence is observed to be convergent as M increases, then its limit

point is an optimal solution to the original non-standard optimal control Problem B.

Remark 2. *We refer to an approximation in which accumulation points of a sequence*

of optimal solutions to the approximate problem are optimal solutions to the original

problem as a zeroth order *consistent approximation. We contrast this condition to that*

of Polak [61], Section 3.3, which in addition requires a condition on stationary points.

We note that the consistency property in Theorem 3.3.2 differs from the consistency results in Ref. [29, 41, 42, 61] because the discretization occurs in the parameter space instead of the time domain. This results in a sequence of standard optimal control problems which can be further approximated using existing time discretization schemes [29, 41, 79].

Note that Theorem 3.3.2 does not ensure the existence of an accumulation point. However, using a generalized Helly's Selection Theorem from Ref. [19], we can guarantee the existence of a convergent subsequence for a certain class of controls.

Definition 5. *[19] Let (Y, d) be a metric space and $h : [0, 1] \mapsto Y$. A function h is of bounded variation if there exists $B > 0$ such that for any partition $\pi, 0 \leq t_0 < t_1 < \ldots < t_n < t_{n+1} \leq 1$, we have $\sum_{i=0}^{n} d(h(t_{i+1}, h(t_i)) < B$. The variation of h is defined as*

$$V_h = \sup_{\pi} \sum_{i=0}^{n} d(h(t_{i+1}), h(t_i))$$

We say family H of functions is of uniformly bounded variation if there exists a $C > 0$ such that for each $h \in H$, we have $h : [0, 1] \mapsto Y$ and $V_h < C$.

Corollary 3.3.3. *Suppose Assumptions 1-8 hold, and in addition there exists $V \in \mathcal{N}_\infty^\#$ and a set of optimal solutions $\{x_M^*, u_M^*\}_{M \in V}$ to Problem B^M, such that $\{u_M^*\}_{M \in V}$ have uniformly bounded variation. Then there exists $V' \subseteq V$ such that $\lim_{M \in V'} \{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$ for some $\{x^\infty, u^\infty\} \in \mathcal{A}$.*

Sketch of Proof: Because $\dot{x} = f(x, u)$ and f is bounded on $X \times U$, $\{x_M^*\}_{M \in V}$ is of uniformly bounded variation on X. Therefore $\{x_M^*, u_M^*\}$ is of uniformly bounded

variation on $X \times U$. Furthermore, $\{x_M^*(t), u_M^*(t)\}_{M \in V}$ is relatively compact, as it is a subset of a compact space. Therefore by the generalization of Helly's Selection Theorem [19], there exists a $V' \subset V$ such that $\lim_{M \in V'}\{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$. $\qquad \square$

It is known that for constrained optimal control problems, the optimal control often belongs to the class of bang-bang controllers, and are piecewise differentiable. If the first derivatives and number of jump discontinuities are bounded, the controls will satisfy the hypothesis in Corollary 3.3.3. Therefore the existence of an accumulation point of optimal pairs to Problem B^M can be guaranteed in this case. From Theorem 3.3.2, it is known that this accumulation point is an optimal pair to Problem B.

Remark 3. *The reader may notice that we have used pointwise convergence of the state and control to establish the optimality result instead of a weaker condition such as $\mathcal{L}^p$ convergence. The result of Theorem 3.3.2 can be established using the $\mathcal{L}^1$ convergence of the state and control, therefore it will hold under this weaker assumption. However, in this work we focus on the stronger condition of pointwise convergence, as it is necessary to establish the Hamiltonian minimization condition considered in Section 3.4.*

Example 1. We demonstrate the convergence properties on a simplified uncertain optimal control problem for which an analytic optimal solution can be derived. Consider the problem of minimizing the cost functional

$$J = \int_\Omega \Big(\int_0^1 \sum_{k=1}^K \big[(x_k(t) - \omega_k)^2 + u_k^2(t)\big] dt \Big) p(\omega) d\omega,$$

where $\omega^T = [\omega_1, \ldots, \omega_K]^T$ with ω_k, $k = 1, 2, \ldots, K$, be independent random variables with joint distribution $p(\omega)$, subject to dynamics $\dot{x}_k(t) = u_k(t)$, and initial condition

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$x_k(0) = 0,\ k = 1, 2, \ldots, K$. In optimal search context, this objective function can represent the K-dimensional distance to a stationary target at position $(\omega_1, \omega_2, \ldots, \omega_K)$ with a penalty function $u_k^2(t)$ intended to keep the control within reasonable bounds.

For parameter ω_k, we can assign a set of nodes $\{\omega_{k,i}^M\}_{i=1}^M$ and weights $\{\alpha_{k,i}^M\}_{i=1}^M$ to approximate the integral over the parameter space based on any numerical integration scheme that satisfies Assumption 8. Remember that the random variables ω_k are independently distributed. We define p_k to be the corresponding probability densities, and introduce the following notations

$$c_k^M = \sum_{i=1}^M p_k(\omega_{k,i}^M)\alpha_{k,i}^M,\ c_{-k}^M = \prod_{j \neq k} c_j^M,\ c^M = \prod_k c_k^M.$$

Using these notations, the discretized uncertain optimal control Problem B^M can be written as: minimizing

$$\sum_{k=1}^K c_{-k}^M \sum_{i=1}^M \left[\int_0^1 (x_k(t) - \omega_{k,i}^M)^2 + u_k^2(t)dt \right] p_k(\omega_{k,i}^M)\alpha_{k,i}^M$$

subject to $\dot{x}_k(t) = u_k(t),\ x_k(0) = 0,\ k = 1, 2, \ldots, K$. This is a standard quadratic linear optimal control problem, which can be solving analytically using the Pontryagin Minimum Principle. The closed-form optimal trajectory and control are given by

$$x_{k,M}^*(t) = \frac{1}{c^M} \sum_{i=1}^M \omega_{k,i}^M p_k(\omega_{k,i}^M)\alpha_{k,i}^M \left(1 - \frac{e^t + e^{2-t}}{1 + e^2}\right)$$

$$u_{k,M(t)}^* = -\frac{1}{c^M} \sum_{i=1}^M \omega_{k,i}^M p_k(\omega_{k,i}^M)\alpha_{k,i}^M \frac{e^t - e^{2-t}}{1 + e^2}.$$

From the definition of c_k^M, c_{-k}^M and the convergence of the numerical scheme, we have

$\lim_{M \to \infty} c_k^M = \lim_{M \to \infty} c_{-k}^M = \lim_{M \to \infty} c^M = 1$, and

$$\lim_{M \to \infty} \sum_{i=1}^{M} \omega_{k,i}^M p_k(\omega_{k,i}^M)\alpha_{k,i}^M = \overline{\omega}_k,$$

where $\overline{\omega}_k = \int_\Omega \omega_k p(\omega)d\omega$. Therefore, the optimal solution of Problem B^M, $\{x_{k,M}^*, u_{k,M}^*\}$, has a limit point as $M \to \infty$, given by

$$x_k^*(t) = \lim_{M \to \infty} x_{k,M}^*(t) = \overline{\omega}_k\left(1 - \frac{e^t + e^{2-t}}{1 + e^2}\right), \tag{3.5}$$

$$u_k^*(t) = \lim_{M \to \infty} u_{k,M}^*(t) = -\overline{\omega}_k\frac{e^t - e^{2-t}}{1 + e^2}. \tag{3.6}$$

According to Theorem 3.3.2, it can be concluded that x_k^* is the optimal trajectory for the considered non-standard optimal control problem and u_k^* is the corresponding optimal control. In this example, because the solution to the approximate optimal control Problem B^M can be given in closed form, it is possible to demonstrate the pointwise convergence of the approximate state and control. In scenarios where the approximate optimal control problem cannot be solved analytically, the pointwise convergence property required in Theorem 3.3.2 can be verified numerically.

3.4 Convergence in the Adjoint Variables

In this section we analyze the convergence of the adjoint variables and Hamiltonian of Problem B^M and provide a necessary condition which is satisfied by accumulation points of a sequence of optimal solutions. In Section 4.1 we showed that by introducing an auxiliary vector $\zeta_M(t) = [z(t, \omega_1^M), \ldots, z(t, \omega_M^M)]^T$, where z is given by (3.3), Problem B^M can be reformulated as a standard optimal control problem with a Bolza

cost. It therefore admits the Hamiltonian $\mathbf{H}^M : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^{n_u} \times [0,1] \mapsto \mathbb{R}$ given by

$$
\begin{aligned}
\mathbf{H}^M(x, \lambda, \zeta_M, \eta_M, u, t) =& \dot{x}^T \lambda + [\dot{\zeta}_M]^T \eta_M \\
=& [f(x, u)]^T \lambda + \sum_{i=1}^{M} [r(x, u, t, \omega_i^M)]^T \eta_{M,i},
\end{aligned}
\tag{3.7}
$$

where λ and η_M are the adjoint variables (costates) corresponding to x and ζ_M respectively. By Pontryagin Minimum Principle [33], if $\{x_M^*, u_M^*\}$ is an optimal solution to Problem B^M, then there exist absolutely continuous costates λ_M^* and η_M^* such that the following conditions hold for almost every $t \in [0, 1]$:

$$
\begin{aligned}
u_M^*(t) &\in \arg \min_{u \in U} \mathbf{H}^M \big(x_M^*(t), \lambda_M^*(t), \zeta_M^*(t), \eta_M^*(t), u, t\big), \\
\dot{\lambda}_M^*(t) &= -\frac{\partial \mathbf{H}^M}{\partial x_M^*} \big(x_M^*(t), \lambda_M^*(t), \zeta_M^*(t), \eta_M^*(t), u_M^*(t), t\big), \\
\dot{\eta}_M^*(t) &= -\frac{\partial \mathbf{H}^M}{\partial \zeta_M^*} \big(x_M^*(t), \lambda_M^*(t), \zeta_M^*(t), \eta_M^*(t), u_M^*(t), t\big).
\end{aligned}
$$

Moreover, the costates satisfy the transversality conditions

$$
\begin{aligned}
\lambda_M^*(1) &= \frac{\partial \hat{J}^M}{\partial x} \big(x_M^*(1), \zeta_M^*(1)\big), \\
\eta_M^*(1) &= \frac{\partial \hat{J}^M}{\partial \zeta_M} \big(x_M^*(1), \zeta_M^*(1)\big).
\end{aligned}
\tag{3.8}
$$

Note that the Hamiltonian (3.7) and adjoint equation of η_M^* lead to

$$
\dot{\eta}_M^* = -\frac{\mathbf{H}^M}{\partial \zeta_M^*} = 0.
$$

Therefore, for all $t \in [0, 1]$, we have $\eta_{M,i}^*(t) = \eta_{M,i}^*(1)$. Thus, from the transversality condition (3.8) and the objective function (3.4), we have, for $i = 1, \ldots M$,

$$
\eta_{M,i}^*(t) = \nabla G \big(\zeta_{M,i}^*(1)\big) p(\omega_i^M) \alpha_i^M.
\tag{3.9}
$$

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The value, $\zeta_{M,i}(1)$, is given by

$$\zeta_{M,i}(1) = z(1, \omega_i^M) = \int_0^1 r(x(t), u(t), t, \omega_i^M) dt.$$

Let Z be the set of all functions from $[0,1] \times \Omega \to \mathbb{R}^K$. We can therefore define an equivalent form of the Hamiltonian, from (3.7) and (3.9), so that $\bar{\mathbf{H}}^M : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times Z \times [0,1]$ is given by

$$\bar{\mathbf{H}}^M(x, \lambda, u, z, t) = [f(x, u)]^T \lambda + \sum_{i=1}^{M} r(x, u, t, \omega_i^M) \nabla G\Big(z(1, \omega_i^M)\Big) p(\omega_i^M) \alpha_i^M. \quad (3.10)$$

From this form of the Hamiltonian and the costate dynamics we get the following adjoint equation for λ_M^*

$$\dot{\lambda}_M^*(t) = - \left[f_x(x_M^*(t), u_M^*(t)) \right]^T \lambda_M^*(t)$$
$$- \sum_{i=1}^{M} \left[r_x \left(x_M^*(t), u_M^*(t), t, \omega_i^M \right) \right]^T \nabla G \left(z_M^*(1, \omega_i^M) \right) p(\omega_i^M) \alpha_i^M, \quad (3.11)$$

where z_M^* is the solution to (3.3) for the optimal pair $\{x_M^*, u_M^*\}$ and the final value is given by the transversality condition:

$$\lambda_M^*(1) = \sum_{i=1}^{M} \nabla_x F(x_M^*(1), \omega_i^M) p(\omega_i^M) \alpha_i^M. \quad (3.12)$$

Now, the necessary condition can be reformulated as:

Necessary Condition of Problem $\mathbf{B}^M$: Suppose that $\{x_M^*, u_M^*\}$ is an optimal pair for Problem B^M. Then u_M^* must satisfy

$$u_M^*(t) \in \arg\min_{u \in U} \bar{\mathbf{H}}^M(x_M^*(t), \lambda_M^*(t), u, z_M^*, t) \quad (3.13)$$

for almost every $t \in [0,1]$, where $\bar{\mathbf{H}}^M$ is given by (3.10), and λ_M^* is given by (3.11)-(3.12) and z_M^* is the solution to (3.3) for the pair $\{x_M^*, u_M^*\}$.

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We now demonstrate the convergence of the adjoint states λ_M^* and Hamiltonians $\bar{\mathbf{H}}^M$. For this purpose, let λ^∞ be the solution of the initial value problem

$$\dot{\lambda}^\infty(t) = - \int_\Omega \left[r_x \left(x^\infty(t), u^\infty(t), t, \omega \right) \right]^T \nabla G \left(z^\infty(1, \omega) \right) p(\omega) d\omega$$

$$- \left[f_x(x^\infty(t), u^\infty(t)) \right]^T \lambda^\infty(t), \tag{3.14}$$

$$\lambda^\infty(1) = \int_\Omega \nabla_x F \left(x^\infty(1), \omega \right) p(\omega) d\omega, \tag{3.15}$$

where z^∞ is the solution to (3.3) for the pair $\{x^\infty, u^\infty\}$. Furthermore, we define the Hamiltonian of Problem B as $\mathbf{H} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times Z \times [0, 1]$ such that

$$\mathbf{H}(x, \lambda, u, z, t) = [f(x, u)]^T \lambda(t) + \int_\Omega [r(x, u, t, \omega)]^T \nabla G \left(z(1, \omega) \right) p(\omega) d\omega. \tag{3.16}$$

Remark 4. *As opposed to the Hamiltonian used in the optimal control of distributed parameter systems, the Hamiltonian defined in (3.16) does not explicitly depend on the unknown parameter. This is because the optimal control of Problem B is not a function of the unknown parameter, which is different from the distributed parameter problem.*

Theorem 3.4.1. *Suppose Assumptions 1-8 hold. Let $V \in \mathcal{N}_\infty^\#$ and let $\{x_M^*, u_M^*\}_{M \in V}$ be a set of optimal solutions to Problem B^M such that $\lim_{M \in V} \{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$. Let λ_M^* be the solutions to (3.11)-(3.12), and λ^∞ be the solution to (3.14)-(3.15). Then for every $t \in [0, 1]$*

$$\lim_{M \in V} \lambda_M^*(t) = \lambda^\infty(t).$$

Moreover, for $\bar{\mathbf{H}}^M$ and $\mathbf{H}$ defined in (3.10) and (3.16),

$$\lim_{M \in V} \bar{\mathbf{H}}^M(x_M^*(t), \lambda_M^*(t), u_M^*(t), z_M^*, t)$$

$$= \mathbf{H}(x^\infty(t), \lambda^\infty(t), u^\infty(t), z^\infty, t).$$

Proof. Define λ'_M to be the solution of system (3.11) with final condition (3.15). Consider the difference

$$
\begin{aligned}
\lambda'_M(t) - \lambda^\infty(t) &= \int_t^1 \dot{\lambda}'_M(s)ds - \int_t^1 \dot{\lambda}^\infty(s)ds \qquad\qquad (3.17)\\
&= \int_t^1 \Big[f_x^\infty(s)^T \lambda^\infty(s) - f_x^M(s)^T \lambda'_M(s) \Big]\\
&\quad + \Big[\int_\Omega \Gamma^\infty(t,\omega)p(\omega)d\omega - \sum_{i=1}^M \Gamma_M^*(t,\omega_i^M)p(\omega_i^M)\alpha_i^M \Big] ds
\end{aligned}
$$

where, for notational simplicity, we have defined

$$
f_x^\infty(t) = f_x(x^\infty(t), u^\infty(s)),
$$

$$
f_x^M(t) = f_x(x_M^*(t), u_M^*(s)),
$$

$$
\Gamma^\infty(t,\omega) = [r_x(x^\infty(t), u^\infty(t), t, \omega)]^T \nabla G(z^\infty(1,\omega)),
$$

$$
\Gamma_M^*(t,\omega) = [r_x(x_M^*(t), u_M^*(t), t, \omega)]^T \nabla G(z_M^*(1,\omega)).
$$

By Assumption 3, f_x is continuous on the compact set $X \times U$; therefore, $f_x(x_M^*(t), u_M^*(t))$ is uniformly bounded. From (3.14) it is seen that λ^∞ is the solution to a system of linear differential equations with bounded coefficients, thus is bounded on the compact domain $[0,1]$. By the dominated convergence theorem

$$
\lim_{M \in V} \int_0^1 f_x^M(t)^T \lambda^\infty(t)dt = \int_0^1 f_x^\infty(t)^T \lambda^\infty(t)dt.
$$

Similarly, by Assumption 3-8 and Remark 1, it can be shown that

$$
\lim_{M \in V} \sum_{i=1}^M \Gamma_M^*(t,\omega_i^M)p(\omega_i^M)\alpha_i^M = \int_\Omega \Gamma^\infty(t,\omega)p(\omega)d\omega.
$$

From these limits, for each $\epsilon > 0$, let $M' \in \mathbb{N}$ be such that, for every $M \in V$ with

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$M > M'$,

$$\int_0^1 \|f_x^M(t)^T \lambda^\infty(t) - f_x^\infty(t)^T \lambda^\infty(t)\| dt < \epsilon \qquad (3.18)$$

$$\int_0^1 \left\| \int_\Omega \Gamma^\infty(t,\omega) p(\omega) d\omega - \sum_{i=1}^M \Gamma_M^*(t,\omega_i^M) p(\omega_i^M) \alpha_i^M \right\| dt < \epsilon. \qquad (3.19)$$

Therefore, by (3.17), (3.18), and (3.19),

$$\left\| \lambda^\infty(t) - \lambda_M'(t) \right\| < \int_t^1 \left\| f_x^M(s)^T \lambda_M'(s) - f_x^\infty(s)^T \lambda^\infty(s) \right\| ds + \epsilon$$

$$\leq \int_t^1 \left\| f_x^M(s)^T \lambda_M'(s) - f_x^M(s)^T \lambda^\infty(s) \right\| ds$$

$$+ \int_t^1 \left\| f_x^M(s)^T \lambda^\infty(s) - f_x^\infty(s)^T \lambda^\infty(s) \right\| ds + \epsilon$$

$$\leq \int_t^1 \left\| f_x^M(s)^T \lambda_M'(s) - f_x^M(s)^T \lambda^\infty(s) \right\| ds + 2\epsilon.$$

Applying Gronwall's inequality gives

$$\|\lambda^\infty(t) - \lambda_M'(t)\| \leq 2\epsilon \int_t^1 \exp \left\| f_x^M(s) \right\| ds.$$

The function in the integral is uniformly bounded in M, and for any $\epsilon > 0$ we can find

an S such that the statement is valid for each $M \in V, M > S$, thus

$$\lim_{M \in V} \lambda_M'(t) = \lambda^\infty(t).$$

Recall that the final conditions, $\lambda_M^*(1)$ and $\lambda_M'(1)$ are given by (3.12) and (3.15). By

Assumption 8 and the continuous dependence of dynamical systems on the initial con-

dition, combined with the convergences $x_M^*(1) \to x^\infty(1)$ and $\lambda_M'(t) \to \lambda^\infty(t)$, for each

$\epsilon > 0, t \in [0,1]$ there exists $N \in \mathbb{N}$ such that for each $M > N, M \in V$, the following

conditions hold:

$$\left\| \lambda_M^*(t) - \lambda_M'(t) \right\| < \frac{\epsilon}{2}, \quad \left\| \lambda_M'(t) - \lambda^\infty(t) \right\| < \frac{\epsilon}{2}.$$

Therefore

$$\lim_{M \in V} \lambda_M^*(t) = \lambda^\infty(t).$$

The proof of the convergence of the Hamiltonians follows a similar argument. $\qquad\square$

Given the convergence of the adjoint variables and Hamiltonians, we can now show that if the solutions to Problem B^M have an accumulation point, this accumulation point must minimize the Hamiltonian for Problem B.

Theorem 3.4.2. *Suppose Assumptions 1-8 hold. Let $V \in \mathcal{N}_\infty^\#$ and let $\{x_M^*, u_M^*\}$ be a sequence of optimal pairs to Problem B^M such that $\lim_{M \in V}\{x_M^*, u_M^*\} = \{x^\infty, u^\infty\}$. Then there exists an absolutely continuous costate trajectory λ^∞ satisfying (3.14)-(3.15) such that the following holds for almost every $t \in [0,1]$:*

$$u^\infty(t) \in \arg \min_{u \in U} \mathbf{H}(x^\infty(t), \lambda^\infty(t), u, z^\infty, t) \qquad (3.20)$$

where $\mathbf{H}$ is given by (3.16) and z^∞ is the solution to (3.3) for the pair $\{x^\infty, u^\infty\}$.

Proof. From Theorem 3.4.1, $\lim_{M \in V} \lambda_M^* = \lambda^\infty$ and

$$\lim_{M \in V} \bar{\mathbf{H}}^M(x_M^*(t), \lambda_M^*(t), u, z_M^*, t)$$

$$= \mathbf{H}(x^\infty(t), \lambda^\infty(t), u, z_M^*, t).$$

Then for any admissable $u \in U$ and each $t \in [0,1]$

$$\mathbf{H}(x^\infty(t), \lambda^\infty(t), u^\infty(t), z^\infty, t) = \lim_{M \in V} \bar{\mathbf{H}}^M(x_M^*(t), \lambda_M^*(t), u_M^*(t), z_M^*, t)$$

$$\leq \lim_{M \in V} \bar{\mathbf{H}}^M(x_M^*(t), \lambda_M^*(t), u, z_M^*, t)$$

$$= \mathbf{H}(x^\infty(t), \lambda^\infty(t), u, z^\infty, t).$$

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□

In the previous section, Theorem 3.3.2 shows that an accumulation point of the set of optimal pairs to Problem B^M is an optimal solution of Problem B. Theorem 3.4.2 further provides necessary conditions that such an accumulation point must satisfy. Such results can be applied to verify the optimality of the computed solution. It can also be used to develop algorithms for Problem B by solving the necessary conditions as demonstrated in the following example.

Revisit of Example 1. In the previous section, the analytic optimal solution of Example 1 was obtained by an application of Theorem 3.3.2. Now we show that Theorem 3.4.2 provides an alternative way to solve this example problem. First note that in this example, $G(z) = z$, so that $\nabla G(z(1, \omega)) = 1$. Then from (3.16) the Hamiltonian of this problem is given by:

$$\mathbf{H}(x(t), \lambda(t), z, u(t), t) = \lambda^T(t)u(t) + \sum_{k=1}^{K} \left(x_k^2(t) + u_k^2(t) - 2x_k(t)\overline{\omega}_k + \overline{\omega_k^2} \right),$$

where $\overline{\omega}_k = \int_\Omega \omega_k d\omega$, $\overline{\omega_k^2} = \int_\Omega \omega_k^2 d\omega$. Here we use the independence of the random variables ω_k to evaluate the integral over Ω. The costate, $\lambda = [\lambda_1, \ldots, \lambda_K]^T$, satisfies adjoint equation

$$\dot{\lambda}_k(t) = -2x_k(t) + 2\overline{\omega}_k, \tag{3.21}$$

$$\lambda_k(1) = 0, \ k = 1, \ldots, K.$$

Because the system is unconstrained, the Hamiltonian minimization condition in The-

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orem 3.4.2 requires

$$\frac{\partial \mathbf{H}}{\partial u_k} = \lambda_k(t) + 2u_k(t) = 0, \tag{3.22}$$

for $k = 1, \ldots, K$. Equation (3.21), (3.22), together with dynamics, results in a boundary value problem

$$\begin{bmatrix} \dot{x}_k(t) \\ \dot{\lambda}_k(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 2\overline{\omega}_k \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_k(t) \\ \lambda_k(t) \end{bmatrix},$$

$$x_k(0) = 0, \ \lambda_k(1) = 0,$$

that can be solved to yield the same optimal solution as shown in (3.5)-(3.6). Therefore in this scenario, the necessary condition of Theorem 3.4.2 can be used to determine the closed form solution to Problem B.

Similar to Corollary 3.3.3, the following result can be established to ensure the existence of a solution satisfying the condition of Theorem 3.4.2.

Corollary 3.4.3. *Suppose Assumptions 1-8 hold, and in addition there exists $V \in \mathcal{N}_\infty^{\#}$ and a set of optimal solutions $\{x_M^*, u_M^*\}_{M \in V}$ to Problem B^M, such that $\{u_M^*\}_{M \in V}$ have uniformly bounded variation. Then there exists an optimal solution, $\{x^\infty, u^\infty\}$, to Problem B and a costate, λ^∞, satisfying condition (3.14), (3.15) and (3.20).*

Proof. The corollary follows direction from Corollary 3.3.3 and Theorem 3.4.2. □

3.5   Application on Optimal Search

In this section, we apply the results of the previous sections to an optimal search problem inspired by a real-world scenario. The example, taken from [22] and [23], considers a surface vessel attempting to detect a hostile target with sonar. The target travels towards a friendly ship, called the "high value unit" or "HVU." The objective of the problem is to find a search path that maximizes the chance of detecting the target, before the target reaches the "HVU."

The searcher is modeled as a Dubin's vehicle with dynamics

$$\dot{x}_1(t) = v \cos x_3(t),$$

$$\dot{x}_2(t) = v \sin x_3(t), \qquad (3.23)$$

$$\dot{x}_3(t) = u(t),$$

where (x_1, x_2) represents the position of the searcher and x_3 is the heading angle. The forward velocity is set to be a constant $v = 150$. The control, u, is the turning rate of the vehicle that satisfies $|u(t)| \leq 50$ for all $t \in [0, 1]$. In the scenario we consider, the HVU travels in the positive x_2 direction at a constant speed of 25, and the starting location of the HVU is $(35, 0)$. The initial state of the searcher is given by $(x_1(0), x_2(0), x_3(0)) = (35, 0, \frac{\pi}{6})$. We assume the trajectory of the target is conditionally deterministic, with starting x_1 coordinate fixed at 70 and x_2 coordinate distributed in the domain $[0, 100]$ according to a Beta distribution. That is, the starting location of the target is given by $y(0, \omega) = (70, \omega)$ for $\omega \in [0, 100]$ and $p(\omega) = p_{4,2}(\omega/100)$, where $p_{\alpha,\beta}$ is the probability density of a Beta(α, β) distribution. For a given starting location, the target moves to

intercept the HVU with a trajectory determined by the algorithm specified in Ref. [26].

The uncertain optimal control problem is then to determine a control input u which will minimize the probability of not detecting the target subject to the searcher dynamics (3.23), control constraint, and given initial conditions. As explained in Section 1.1, the probability of non-detection can be modeled as

$$J = \int_0^{100} \exp\Big(-\int_0^1 \tilde{r}(x(t), y(t,\omega))dt \Big) p(\omega)d\omega,$$

where $\tilde{r}$ is the instantaneous rate of detection. The specific form of the detection rate function depends on the sensor. In this example we use the Poisson scan model:

$$\tilde{r}(x(t), y(t,\omega)) = \beta\, \Phi\Big(\frac{F^k - D\, \|x(t) - y(t,\omega)\|^2 - b}{\sigma} \Big),$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function, $\|x(t) - y(t,\omega)\|$ is the Euclidean distance between the searcher and the target, β is the scan opportunity rate, F^k is the so-called "figure of merit" (a sonar characteristic), and σ reflects the variability in the "signal excess". In the simulation we use the values $\beta = 1.9$, $F^k = 120$, $b = 20$, $D = 0.45$, and $\sigma = 150$.

The proposed computational framework is applied to this search problem with a LGL quadrature discretization in the parameter space with 42 nodes. Applying this discretization results in a standard optimal control problem which is solved using a pseudospectral discretization scheme in the time domain [30, 69]. The NLP package SNOPT [27] is used to calculate the solution to NLP problem produced by this sequence of approximations. This yields a numerical approximation to the optimal trajectory for the searcher.

Figure 3.1 demonstrates the numerical solution obtained by the proposed computational scheme. Snapshots of the searcher and HVU trajectories are shown in Figure 3.1.a–d. For reference, a random sample of target trajectories with the initial starting location subject to a Beta$(4, 2)$ distribution is also shown. Shown in frame $a)$, the searcher moves away from the HVU towards the right boundary $x_1 = 70$, as it is known that the target originates at this line. In frames $b) - d)$, the searcher, knowing that the target is moving to intercept the HVU, tracks the possible target trajectories back towards the HVU, while adjusting its trajectory so as to match velocity to the target. To assess the validity of the numerical solution, we compute costates, λ_i, $i = 1, 2, 3$, according to (3.14)-(3.15) using the numerical solution $\{x, u\}$. Observe that the control u enters into the Hamiltonian only through the linear term $\lambda_3 u$. Therefore, the Hamiltonian minimization condition (3.20) implies that

$$u(t) = \begin{cases} 50, & \text{if } \lambda_3(t) < 0 \\ -50, & \text{if } \lambda_3(t) > 0 \end{cases} \tag{3.24}$$

In other words, optimal control is of bang-bang type where λ_3 is the switching function. As shown in Figure 3.2 the Hamiltonian minimization condition (3.24) is indeed satisfied.

Next we consider a scenario which differs from the previous scenario only in the initial position of the target. In this scenario, the initial condition of the target is modeled by a mixture of beta distributions, that is, $p(\omega) = p_{12,1}(\omega/100) + p_{1,12}(\omega/100)$, where $p_{\alpha,\beta}$ is the probability density of a Beta(α, β) distribution. In this model, at

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Figure 3.1: Computed optimal trajectory for a searcher attempting to detect a target which is moving to intercept a high-value unit (HVU). The starting location of target is unknown to the searcher and modeled by a Beta distribution. Arrows indicate the orientation of the searcher, target and HVU trajectories. For reference, a random sample of target trajectories is shown, where the initial starting location is determined by a $\text{Beta}(4,2)$ distribution. The trajectory is computed using an LGL quadrature discretization in the parameter space and an LGL-pseudospectral method in the time domain, together with the NLP package SNOPT.
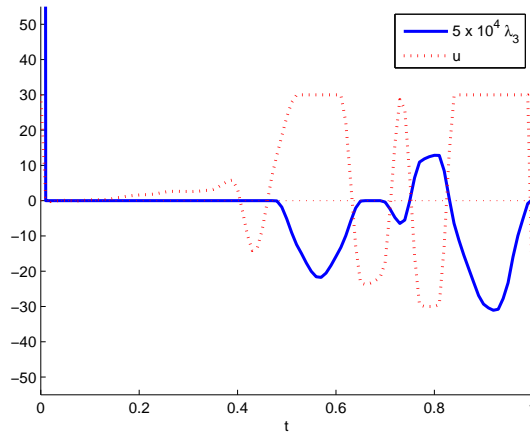
Figure 3.2: The optimal control for the optimal search problem is of a bang-bang type. This figure shows the switching function λ_3 and optimal control u.

the initial time the target is likely to be near $(70, 0)$ or $(70, 100)$, but significantly less likely to be near $(70, 50)$. From the computed optimal searcher trajectory shown in Figure 3.3, it is clear that the optimal behavior of the searcher changes depending on the information the searcher has about the starting location of the target. The searcher knows possible target trajectories are very likely to be in one of two groups, one originating near the bottom of the frame and one near the top of the frame. In Fig.3.3.a the searcher, knowing that the target is unlikely to be near the middle of the frame, moves towards the right boundary $x_1 = 70$, but nearer the bottom of the frame. In Fig.3.3.b-c the searcher tracks the possible target trajectories back towards the HVU while adjusting its trajectory to match velocity at the target. However, due to the decreasing nature of the detection function, this strategy has diminishing returns. In Fig.3.3.d, the searcher leaves the bottom group of possible target trajectories and moves upwards in an attempt to detect the second group of possible target trajectories.
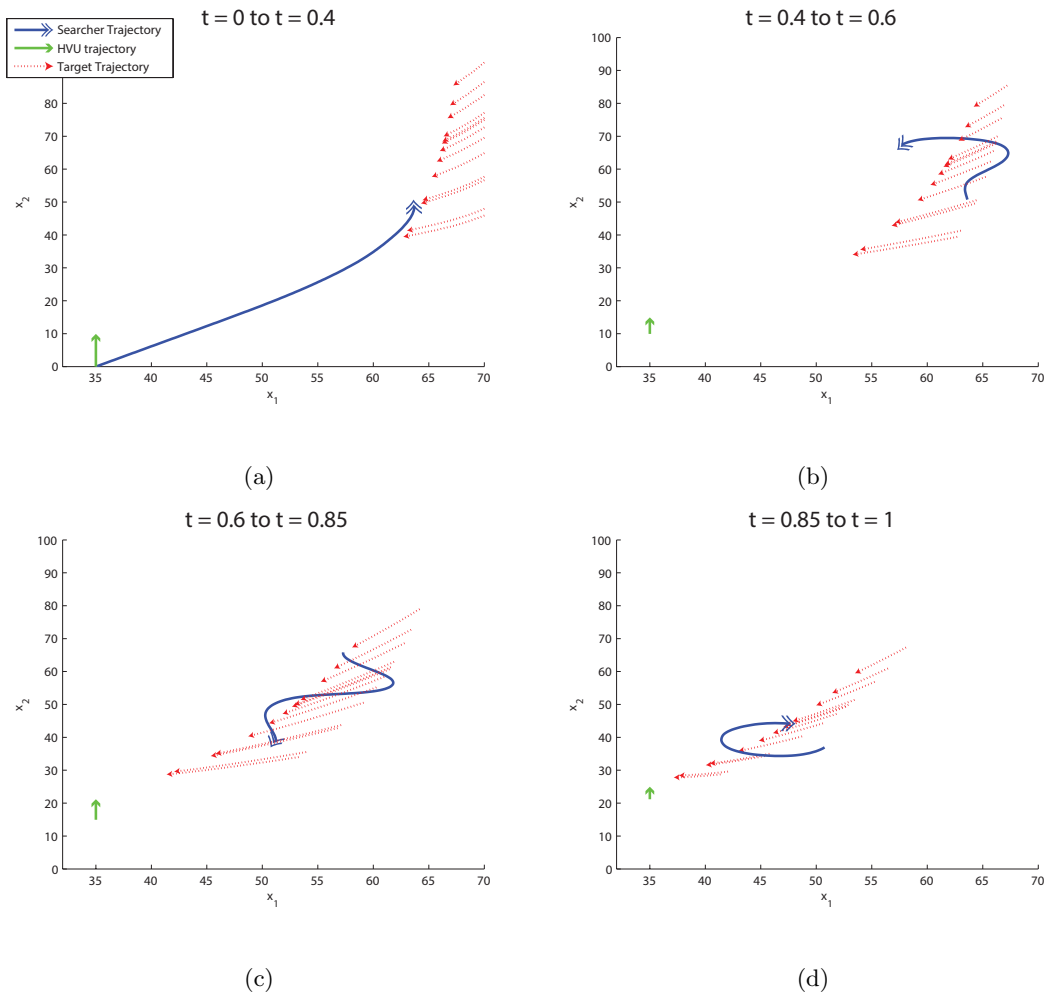
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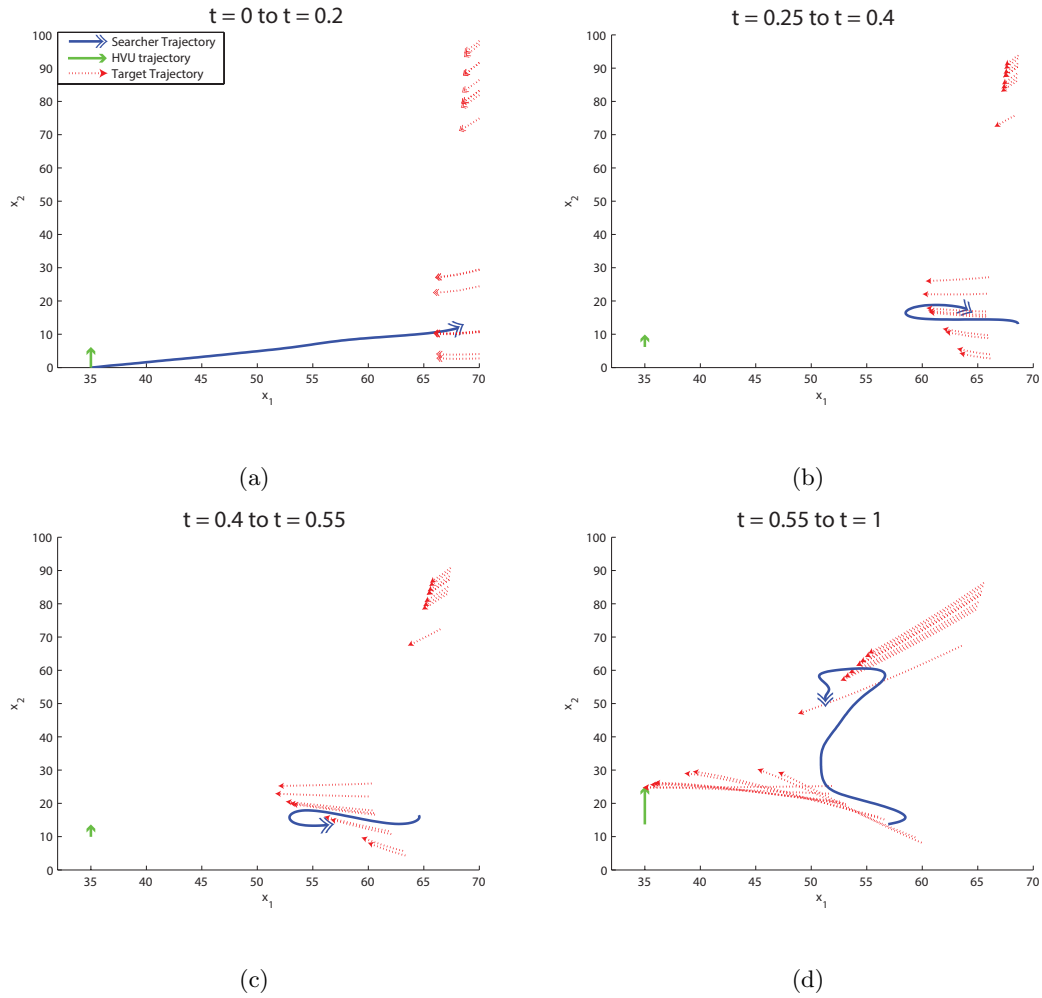
Figure 3.3: Computed optimal trajectory for a searcher attempting to detect a target which is moving to intercept a high-value unit (HVU). The starting location of target is unknown to the searcher and modeled by a mixture of Beta distributions. Arrows indicate the orientation of the searcher, target and HVU trajectories. For reference, a random sample of target trajectories is shown, where the initial starting location is determined by a Beta$(4, 2)$ distribution. The trajectory is computed using an LGL quadrature discretization in the parameter space and an LGL-pseudospectral method in the time domain, together with the NLP package SNOPT.

3.6 An Extension to Agents with Uncertain Dynamics

In this section we extend the numerical methods and necessary conditions presented in previous sections to problems which incorporate parameter uncertainty in the agent dynamics and initial state. We compare this work to that of Ref. [73, 75] which focus on an LGL-quadrature approximation of the parameter space. We extend these results to demonstrate that any convergent quadrature scheme can be used to approximate the parameter space, which is advantageous because the convergence properties of the approximated state variables depend on the scheme chosen and the probability distribution of the stochastic parameters [6].

Problem C. Determine the control function $u \in L_\infty([0,1]; \mathbb{R}^{n_u})$ that minimizes the cost functional

$$J[x,u] = \int_\Omega \left[F\left(x(1,\omega),\omega\right) + \int_0^1 r(x(t,\omega), u(t), t, \omega)dt \right] d\omega \tag{3.25}$$

subject to the dynamics

$$\dot{x}(t,\omega) = f(x(t,\omega), u(t), \omega), \tag{3.26}$$

initial condition $x(0,\omega) = x_0(\omega)$, and the control constraint $g(u(t)) \leq 0$ for all $t \in [0,1]$.

In Problem C, $L_\infty([0,1]; \mathbb{R}^{n_u})$ is the set of all essentially bounded functions, $x : [0,1] \times \Omega \mapsto \mathbb{R}^{n_x}$ and $r : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^1 \times \mathbb{R}^{n_\omega} \mapsto \mathbb{R}$. Note that this Problem C formulation differs slightly from that of Chapter 1. Here we have omitted the probability density function p for notational convenience, since it can be included in the functions F and r.

Assumption 5. *The function $g : \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_g}$ is continuous and the set $U = \{\nu \in \mathbb{R}^{n_u} | g(\nu) \leq 0\}$ is compact.*

In a real world scenario the set of allowable controls will be bounded and therefore U, being a closed and bounded set, will be compact.

Assumption 6. *Let $\mathcal{A}$ be the set of feasible controls to Problem C, that is the set of all $u \in L_\infty([0,1]; \mathbb{R}^{n_u})$ such that $u(t) \in U$. There exists a compact set $X \subset \mathbb{R}^{n_x}$ such that for each feasible u and $\omega \in \Omega, t \in [0,1]$, $x(t,\omega) \in X$ where $x(t,\omega) = x_0 + \int_0^t f(x(s,\omega), u(s), \omega)ds$ for all $t \in [0,1]$.*

This assumption essentially requires for all bounded controls that there is no $\omega \in \Omega$ for which the state has a finite escape time. A large class of nonlinear systems satisfy this assumption, for example, input-to-state stable systems and systems for which f is globally Lipschitz or satisfies a linear growth condition.

Assumption 7. *The functions f and r are C^1. The set Ω is compact and $x_0 : \Omega \mapsto \mathbb{R}^{n_x}$ is continuous. Moreover, for the compact sets X and U defined in Assumptions 5-6 and for each $t \in [0,1]$, $\omega \in \Omega$, the Jacobians $r_x(\cdot, \cdot, t, \omega)$ and $f_x(\cdot, \cdot, \omega)$ are Lipschitz on the set $X \times U$, and the corresponding Lipschitz constants L_r and L_f are uniformly bounded in ω and t. The function $F(\cdot, \omega)$ is C^1 on X for all $\omega \in \Omega$; in addition, F and F_x are continuous with respect to ω.*

In this section we demonstrate that a variety of quadrature-based numerical integration schemes can be used to approximate the parameter space of Problem C.

The following assumption about the convergence of the scheme allows methods such as Gaussian quadrature and composite-Simpson to be applied.

Assumption 8. *For each $M \in \mathbb{N}$, there is a set of nodes $\{\omega_i^M\}_{i=1}^M \subset \Omega$ and an associated set of weights $\{\alpha_i^M\}_{i=1}^M \subset \mathbb{R}$, such that for any continuous function $h : \Omega \to \mathbb{R}$,*

$$\int_\Omega h(\omega)d\omega = \lim_{M \to \infty} \sum_{i=1}^M h(\omega_i^M)\alpha_i^M.$$

By selecting a finite number of nodes to approximate the parameter space, we allow the state vector $x : [0,1] \times \Omega \mapsto \mathbb{R}^{n_x}$ to be approximated by the state vector $\bar{X}_M : [0,1] \mapsto \mathbb{R}^{Mn_x}$. We use $\bar{X}_M = [\bar{x}_1^M, \ldots, \bar{x}_M^M]$ to denote the discretized state vector. That is, $\bar{x}_i^M$ is the solution to the dynamical system

$$\dot{\bar{x}}_i^M(t) = f(\bar{x}_i^M(t), u(t), \omega_i^M) \qquad\qquad \bar{x}_i^M(0) = x_0(\omega_i^M). \qquad (3.27)$$

We refer to the state vector $\bar{X}_M$, where the dependence on the parameter ω has been discretized, as the *semi-discretized state*.

Once the numerical scheme is chosen, an approximate objective functional for each $M \in \mathbb{N}$ can be defined by

$$\bar{J}^M[\bar{X}_M, u] = \sum_{i=1}^M \left[F\left(\bar{x}_i^M(1), \omega_i^M\right) + \int_0^1 r(\bar{x}_i^M(t), u(t), t, \omega_i^M)dt \right]\alpha_i^M. \qquad (3.28)$$

Problem $\mathbf{C^M}$. Determine the optimal control $u \in L_\infty([0,1]; \mathbb{R}^{n_u})$, that minimizes the cost functional (3.28) subject to the dynamics (3.27) and the control constraint $g(u(t)) \leq 0$ for all $t \in [0,1]$.

We apply the approach of Sections 3.3 to Problem C. However, because of the process used in the approximation and the inclusion of the stochastic parameter in the

state dynamics, the state space for Problem C^M is of a different dimension than that of Problem C, which introduces new theoretical challenges. Before we introduce the result on the convergence of the state variables, we must define the spaces on which we conduct our analysis, and define a mapping between the two state spaces.

We denote by $W_{1,\infty}([0,1]; \mathbb{R}^{n_x})$ the Sobolev space of all essentially bounded functions with essentially bounded distributional derivatives, which map the interval $[0,1]$ into the space $\mathbb{R}^{n_x}$. We then define the state space $\mathcal{X}$ of Problem C to be the set of all functions $x : [0,1] \times \Omega \mapsto \mathbb{R}^{n_x}$ such that $x(\cdot, \omega) \in W_{1,\infty}([0,1]; \mathbb{R}^{n_x})$ for each $\omega \in \Omega$ and $x(t, \cdot)$ is measurable for each $t \in [0,1]$. Similarly, we define the state space of Problem C^M as $\bar{\mathcal{X}}_M = W_{1,\infty}([0,1]; \mathbb{R}^{Mn_x})$. To create a mapping between these two state spaces we introduce an interpolation scheme which maps the discretized state $\bar{X}_M \in \bar{\mathcal{X}}_M$ to an associated state $x^M \in \mathcal{X}$.

Assumption 9. *For each M, there exists a set of functions $\phi_{M,i} : \Omega \to \mathbb{R}$ such that $\phi_{M,i}(\omega_j^M) = \delta_{i,j}$. For a continuous function $h : \Omega \mapsto \mathbb{R}$, we have $h^M(\omega) = \sum_{i=1}^{M} h(\omega_i^M)\phi_{M,i}(\omega)$ converges uniformly to h.*

Note that this assumption about the convergence properties of the interpolation scheme allows polynomial interpolation schemes such as LGL or Chebyshev to be applied to approximate Problem C.

We define $\Omega^M = \{\omega_1^M, \ldots, \omega_M^M\}$ and for a function $y : \Omega^M \mapsto \mathbb{R}$, we refer to $\gamma : \Omega \mapsto \mathbb{R}$ where $\gamma(\omega) = \sum_{i=1}^{M} y(\omega_i^M)\phi_{M,i}(\omega)$ as the interpolation of y. For a discretized trajectory $\bar{X}_M = [\bar{x}_1^M, \ldots, \bar{x}_M^M]$, we define the interpolation as the trajectory

$\chi_M : [0,1] \times \Omega \mapsto \mathbb{R}^n$ such that $\chi_M(t, \omega) = \sum_{i=1}^{M} \bar{x}_i^M(t)\phi_{M,i}(\omega)$. For a given function $h : \Omega \mapsto \mathbb{R}$, we can create a sequence of functions $y^M : \Omega^M \mapsto \mathbb{R}$ such that the interpolations $\gamma^M \to h$ uniformly in ω by setting $y_M(\omega_i^M) = h(\omega_i^M)$. Therefore each $h : \Omega \mapsto \mathbb{R}$ is a uniform limit of a sequence of interpolating functions for some sequence $y_M : \Omega^M \mapsto \mathbb{R}$. From this property it is clear that for a given $x \in \mathcal{X}$, there exists a sequence $\bar{X}_M \in \bar{\mathcal{X}}_M$ such that the associated interpolation functions $x^M \in \mathcal{X}$ converge uniformly (in ω) to x.

To demonstrate that Problem C^M is an appropriate approximation of Problem C, we must show that given a sequence of optimal solutions $\{\bar{X}_M^*, u_M^*\}$ to Problem C^M, an accumulation point of the interpolations $\{\chi_M^*, u_M^*\}$ is an optimal solution to Problem C. However, it is important to note that if $\bar{X}_M$ is a feasible state for Problem C^M, it is not necessarily true that its interpolation χ_M is a feasible state for Problem C. Therefore when demonstrating the optimality of the accumulation point of such a sequence, it is also necessary to also demonstrate its feasibility.

Theorem 3.6.1. *Suppose Assumptions 5-9 hold. Let $\{(\bar{X}_M^*, u_M^*)\}$ be a sequence of optimal solutions to Problem C^M and let u^∞ be an $\mathcal{L}^2$ accumulation point of u_M^* for a subsequence indexed by $V \in \mathcal{N}_\infty^\#$. Let χ_M^* denote the interpolation of $\bar{X}_M^*$. Then $\chi_M^* \to x^\infty$ uniformly in ω along the subsequence indexed by V, where x^∞ is the solution to (3.26) for the control u^∞. Furthermore (x^∞, u^∞) is an optimal solution to Problem C and $\lim_{M \in V} \bar{J}^M(\bar{X}_M^*, u_M^*) = J(x^\infty, u^\infty)$.*

Proof. For notational simplicity, we denote by x^u the solution to (3.26) for the control

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u, $\bar{X}_M^u$ the solution to (3.27) for the control u and M nodes, and χ_M^u the interpolation of $\bar{X}_M^u$. It is important to note that $\|v\|$, $v \in \mathbb{R}^n$ denotes the Euclidean norm and $\|u\|_2$, $u \in \mathcal{L}^2$ denotes the $\mathcal{L}^2$ norm.

Part 1 We demonstrate the convergence of χ_M^*, $M \in V$. Based on the convergence properties of the dynamical system and the interpolation scheme, the following are true for each $t \in [0, 1]$:

i Due to the uniform convergence of the interpolation scheme (Assumption 9), for every $\epsilon > 0$ there exists an $M_1 \in \mathbb{N}$ and a $\delta > 0$ such that if $h_1, h_2 : \Omega \mapsto \mathbb{R}$ are continuous functions such that $|h_1(\omega) - h_2(\omega)| < \epsilon$ for every $\omega \in \Omega$, then $|h_1^M(\omega) - h_2^M(\omega)| < \epsilon$ for every $\omega \in \Omega$, $M > M_1$.

ii From Ref. [61], Lemma 5.6.5, we have $\|x^{u_1}(t, \omega) - x^{u_2}(t, \omega)\| \le K \|u_1 - u_2\|_2$ for each $\omega \in \Omega$ and some $K \in [1, \infty)$. Therefore if u^∞ is an $\mathcal{L}^2$ accumulation point of $\{u_M^*\}$, the solution to (3.26) for u^∞ is a uniform (in ω) accumulation point of the solution to (3.26) for $\{u_M^*\}$. Therefore there exists $M_2 \in \mathbb{N}$ such that $\left\| x^\infty(t, \omega) - x^{u_{M_2}^*}(t, \omega) \right\| < \epsilon$ for each $\omega \in \Omega$.

iii By Assumption 9, for a fixed control u, the interpolation to the solution of (3.27) for u converges to the solution of (3.26) for u. Therefore there exists an $M_3 \in \mathbb{N}$ such that for each $M > M_3$ we have $\left\| x^{u_{M_2}^*}(t, \omega) - \chi_M^{u_{M_2}^*}(t, \omega) \right\| < \epsilon$ for all $\omega \in \Omega$.

iv Based on the statement in $ii)$, the sequence $x^{u_M^*}$ is uniformly (in ω) Cauchy convergent. Therefore there exists an $M_4 > M_1 \in \mathbb{N}$ such that for each $M > M_4$, we have $\left\| x^{u_{M_2}^*}(t, \omega) - x^{u_M^*}(t, \omega) \right\| < \delta$ for all $\omega \in \Omega$. Based on the statement in $i)$, this

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implies that $\left\| \chi_M^{u_{M_2}^*}(t,\omega) - \chi_M^*(t,\omega) \right\| < \epsilon$ for all $\omega \in \Omega$.

Because $i) - iv)$ hold for every $t \in [0,1]$, for a given $t \in [0,1]$ and $\omega \in \Omega$ we have for each $M > \max\{M_1, M_3, M_4\}$:

$$\|x^\infty(t,\omega) - \chi_M^*(t,\omega)\| = \left\| x^\infty(t,\omega) - x^{u_{M_2}^*}(t,\omega) \right\| + \left\| x^{u_{M_2}^*}(t,\omega) - \chi_M^{u_{M_2}^*}(t,\omega) \right\|$$

$$+ \left\| \chi_M^{u_{M_2}^*}(t,\omega) - \chi_M^*(t,\omega) \right\|$$

$$< 3\epsilon$$

Part 2 We demonstrate that $\lim_{M \in V} \bar{J}^M[\bar{X}_M^*, u_M^*] = J[x^\infty, u^\infty]$. Consider the objective functional J^M which is approximated using the numerical scheme of Assumption 8, that is:

$$J^M[x,u] = \sum_{i=1}^M \left[F\left(x(1,\omega_i^M), \omega_i^M\right) + \int_0^1 r(x(t,\omega_i^M), u(t), t, \omega_i^M) dt \right] \alpha_i^M. \qquad (3.29)$$

Clearly $\{\bar{X}_M^*, u_M^*\}$ is a global minimizer to (3.28) if and only if $\{x^{u_M^*}, u_M^*\}$ is a global minimizer of (3.29), and in this case $\bar{J}^M[\bar{X}_M^*, u_M^*] = J^M[x^{u_M^*}, u_M^*]$. We therefore proceed by showing that if u^∞ is an $\mathcal{L}^2$ accumulation point of u_M^*, then $J^M[x^{u_M^*}, u_M^*] \to J[x^\infty, u^\infty]$.

First note that by Jensen's inequality we have

$$\int_0^1 \|u_M^*(t) - u^\infty(t)\| \, dt = \sqrt{\left[\int_0^1 \|u_M^*(t) - u^\infty(t)\| \, dt \right]^2} \leq \sqrt{\int_0^1 \|u_M^*(t) - u^\infty(t)\|^2 \, dt}$$

$$= \|u_M^* - u^\infty\|_2 \, .$$

From Assumption 7, r is bounded and Lipschitz on $X \times U \times [0,1] \times \Omega$ and G is uniformly continuous on $r(X, U, [0,1], \Omega)$. From the Lipschitz continuity of r, we have,

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for all $\omega \in \Omega$

$$\lim_{M \in V} \int_0^1 \|r(x^{u_M^*}(t, \omega), u_M^*(t), t, \omega) - r(x^\infty(t, \omega), u^\infty(t), t, \omega)\| dt$$

$$\leq \lim_{M \in V} L_r \int_0^1 \|x^{u_M^*}(t, \omega) - x^\infty(t, \omega)\| + \|u_M^*(t) - u^\infty(t)\| dt$$

$$\leq L_r \|u_M^* - u^\infty\|_2 + \lim_{M \in V} L_r \int_0^1 \|x^{u_M^*}(t, \omega) - x^\infty(t, \omega)\| dt = 0$$

Because $x^{u_M^*}(t, \cdot) \to x^\infty(t, \cdot)$ uniformly, this convergence is uniform in ω. Then by the

uniform continuity of F and G, for each $\epsilon > 0$, there must exist $N \in \mathbb{N}$ such that for

each $M \in V$ with $M > N$ the following statements hold for all $\omega \in \Omega$

$$\left\| \int_0^1 r(x^{u_M^*}(t), u_M^*(t), t, \omega) dt - \int_0^1 r(x^\infty(t), u^\infty(t), t, \omega) dt \right\| < \frac{\epsilon}{2},$$

$$\left\| F\left(x^{u_M^*}(1), \omega\right) - F\left(x^\infty(1), \omega\right) \right\| < \frac{\epsilon}{2}.$$

This implies, by the statement in Remark 1, $\lim_{M \in V} J^M[x^{u_M^*}, u_M^*] = J[x^\infty, u^\infty]$. Be-

cause $\bar{J}^M[\bar{X}_M^*, u_M^*] = J^M[x^{u_M^*}, u_M^*]$, we have $\lim_{M \in V} \bar{J}^M[\bar{X}_M^*, u_M^*] = J[x^\infty, u^\infty]$.

Part 3 We demonstrate that $\{x^\infty, u^\infty\}$ is the optimal solution to Problem C. Let u

be an arbitrary feasible control for Problem C. Then, based on the optimality of u_M^*,

$\bar{J}^M(\bar{X}_M^*, u_M^*) \leq \bar{J}^M(\bar{X}_M^u, u)$ for all $M \in V$. Thus

$$J(x^\infty, u^\infty) = \lim_{M \in V} \bar{J}^M(\bar{X}_M^*, u_M^*) \leq \lim_{M \in V} \bar{J}^M(\bar{X}_M^u, u) = J(x^u, u).$$

Therefore (x^{u^∞}, u^∞) is an optimal solution for Problem C, since it produces the mini-

mum cost among all feasible solutions. $\qquad \square$

For the set Ω^M of interpolation nodes used in the approximation of Problem

C, the set of interpolated trajectories $\Xi_M = \{\chi_M \in \mathcal{X} | \bar{X}_M \in \bar{\mathcal{X}}_M\}$ is a linear subspace

of $\mathcal{X}$, so that we can consider Problem C^M as a restriction of Problem C to the linear subspace Ξ_M with approximated objective functional (3.29). Now note that for every $x \in \mathcal{X}$ there exists a sequence $\chi_M \in \Xi_M$ such that $\chi_M \to x$ uniformly in ω, and the approximated objective functional (3.29) epiconverges to (3.25). We therefore compare our convergence result to the consistency result presented in Chapter 4 of Ref. [61], where a similar framework is used to determine the convergence properties of a time-discretization of the standard nonlinear optimal control problem.

The reader may note that it is possible to use Part 2 of the Proof of Theorem 3.6.1 to demonstrate that an accumulation point of a sequence of global minimizers to (3.29) is a global minimizer to (3.25). Therefore it is possible to approximate Problem C by simply approximating the objective functional and solving the problem without approximating the state space and introducing an interpolation scheme. However, the approach taken in Problem C^M is desirable for two reasons. First, by approximating the state space as well as the objective functional, the approximating problem is a standard non-linear optimal control problem to which existing results such as Pontryagin's Minimum Principle [33] can be applied. By addressing the convergence properties of the approximated adjoint variables, we can extend the Covector Mapping Theorem of Ref. [30] to Problem C. Second, the interpolation scheme allows a convenient way to determine the value of the approximated state. Note that without the interpolation scheme, for a given optimal control, if one wishes to know $x_M^*(t, \omega)$ where ω is not a node used in the solution of the approximated problem, it is necessary to solve the ordinary differential equation given by (3.26) for that value of ω. The interpolation scheme

allows one to determine this value and guarantees that the approximation will converge uniformly as the number of nodes used in the approximation scheme increases.

For many applications, the dual variables provide a method to determine the solution of an optimal control problem or a tool to validate a numerically computed solution. For numerical schemes based on direct discretization of the control problem, analyzing the convergence of the dual variables may also lead to insight into the convergence and validity of approximation scheme [30,31]. Because Problem C^M is a standard nonlinear optimal control problem, it admits a dual problem by the Pontryagin Minimum Principle [33]. In this section we address the convergence properties of the dual variables for Problem C^M and the dual Problem $C^{M\lambda}$. Using a weighted Hamiltonian inspired by the Covector Mapping Theorem [30], we demonstrate that for a convergent sequence of optimal solutions to Problem C^M, the corresponding adjoint variables will converge. First we introduce the dual to Problem C:

Problem C^λ. [24]   If (x^*, u^*) is an optimal solution to Problem C, then there exists an absolutely continuous costate vector $\lambda(t, \omega)$ such that

$$\dot{\lambda}^*(t, \omega) = - f_x(x^*(t, \omega), u(t), \omega)\lambda^*(t, \omega) - r_x(x^*(t, \omega), u(t), t, \omega) \tag{3.30}$$

$$\lambda^*(1, \omega) = F_x(x(1, \omega), \omega). \tag{3.31}$$

Furthermore, the optimal control u^* satisfies the equality

$$u^*(t) = \min_{u \in U} \mathbf{H}(x^*, \lambda^*, u, t),$$

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where **H** is given by

$$\mathbf{H}(x, \lambda, u, t) = \int_{\Omega} \left[\dot{x}(t, \omega)^T \lambda(t, \omega) + r(x(t, \omega), u(t), t, \omega) \right] d\omega. \tag{3.32}$$

Next we introduce the dual to Problem C^M. Because Problem C^M is a standard non-linear optimal control problem, it admits a first-order necessary condition in the form of Pontryagin's Minimum Principle. The Hamiltonian and adjoint system are given by:

$$\bar{H}^M(\bar{X}_M, \bar{\Gamma}_M, u, t) = \sum_{i=1}^{M} \left[\dot{\bar{x}}_i^T \bar{\gamma}_i + r(\bar{x}_i, u, t, \omega_i^M) \alpha_i^M \right].$$

Here $\bar{\Gamma}_M = [\bar{\gamma}_1, \dots, \bar{\gamma}_M]$ is the absolutely continuous costate variable satisfying

$$\dot{\bar{\gamma}}_i(t) = - f_x(\bar{x}_i(t), u(t), \omega_i^M) \bar{\gamma}_i - r_x(\bar{x}_i(t), u(t), t, \omega_i^M) \alpha_i^M,$$

$$\bar{\gamma}_i(1) = F_x(\bar{x}_i(1), \omega_i^M) \alpha_i^M.$$

To demonstrate the convergence properties of this system as $M \to \infty$, we introduce the weighted costate vector $\bar{\Lambda}_M = [\bar{\lambda}_1, \dots, \bar{\lambda}_M]$ such that $\alpha_i^M \bar{\lambda}_i = \bar{\gamma}_i$. Using this weighted costate vector, we define

$$\mathbf{H}^M(\bar{X}, \bar{\Lambda}, u, t) = \sum_{i=1}^{M} \left[\dot{\bar{x}}_i(t)^T \bar{\lambda}_i + r(\bar{x}_i(t), u(t), t, \omega_i^M) \right] \alpha_i^M. \tag{3.33}$$

It is clear that $\bar{H}^M(\bar{X}_M, \bar{\Gamma}_M, u, t) = \mathbf{H}^M(\bar{X}_M, \bar{\Lambda}_M, u, t)$. Now note that $\alpha_i^M \dot{\bar{\lambda}}_i^* = \dot{\bar{\gamma}}_i^*$ so that the weighted costate dynamics are given by

$$\dot{\bar{\lambda}}_i^* = - f_x(\bar{x}_i^*(t), u_M^*(t), \omega_i^M) \bar{\lambda}_i^* - r_x(\bar{x}_i^*(t), u_M^*(t), t, \omega_i^M), \tag{3.34}$$

$$\bar{\lambda}_i^*(1) = F_x(\bar{x}_i^*(1), \omega_i^M). \tag{3.35}$$

Using the weighted costate, we define the dual problem to Problem C^M:

Problem C$^{M\lambda}$. If $(\bar{X}_M^*, u_M^*)$ is an optimal solution to Problem C^M, then there exists an absolutely continuous costate vector $\bar{\Lambda}(t) = [\bar{\lambda}_1(t), \ldots, \bar{\lambda}_M(t)]$ given by (3.34)-(3.35). Furthermore, for almost every $t \in [0,1]$, the optimal control u_M^* satisfies the equation

$$u_M^*(t) = \min_{u \in U} \mathbf{H}^M(\bar{X}_M^*, \bar{\Lambda}_M^*, u, t)$$

It is clear that the Hamiltonian $\mathbf{H}^M$ of Problem $C^{M\lambda}$ is the discretization of the Hamiltonian $\mathbf{H}$ of Problem C^λ, and that (3.34-3.35) is the discretization of (3.30-3.31). Therefore Problem $C^{M\lambda}$ is the discretization of Problem C^λ. Unlike the discretization of the time domain, discretization of the parameter space involves no endpoint conditions which must be satisfied or dynamical constraints which must be discretized. The existence of a feasible solution to Problem $C^{M\lambda}$ is not in question, as it is guaranteed by the Pontryagin Minimum Principle. The Covector Mapping Theorem of Ref. [30] can then be extended to the optimal ensemble control framework by the following Theorem, which addresses the convergence of the adjoint variables and Hamiltonians.

Theorem 3.6.2. *Let $\{(\bar{X}_M^*, \bar{\Lambda}_M^*, u_M^*)\}$ be a sequence of solutions to Problem $C^{M\lambda}$, χ_M^* be the interpolation of $\bar{X}_M^*$, $\dot{x}^M$ be the interpolation of $\dot{\bar{X}}_M^*$, ψ_M^* be the interpolation of $\bar{\Lambda}_M^*$. Let u^∞ be an accumulation point of u_M^* and let λ^∞ be the solution to (3.30-3.31) for u^∞. Then ψ_M^* converges uniformly to λ^∞ and $(x^\infty, \lambda^\infty, u^\infty)$ is a solution to Problem C^λ.*

Proof. The proof follows from the same argument as the proof of Theorem 1. $\square$

Chapter 4

A Sample Average Scheme for Approximation of the Uncertain Optimal Control Problem

In Chapter 3 we introduced a scheme for the optimal control of uncertain systems based on a quadrature approximation of the expectation over the parameter space. However, due to the curse of dimensionality, this approach is inherently limited to systems with a low-dimensional parameter space. Indeed, as the number of stochastic parameters increases, the dimension of the approximated optimal control problem increases exponentially, therefore a different numerical method must be used in these cases [6]. This difficulty is inherent to the approximation of dynamical systems with stochastic parameters and other techniques such as polynomial chaos are also computationally expensive for high-dimensional problems. Therefore, in many cases Monte

Carlo simulation is required to approximate an uncertain dynamical system [6].

In this chapter we propose a sample average approximation approach to the uncertain optimal control problem which is applicable for high-dimensional problems. In this method, an independently distributed random draw is taken from the parameter space, and the expectation in the objective functional is approximated by the sample average. We refer to [3,5] for early work on the sample average approximation approach to stochastic optimization, which also provides our foundation. For a treatment of cases in finite dimensions; see [81]. Because the number of nodes sampled does not depend on the dimension of the parameter space, this method does not suffer from the same curse of dimensionality as the previously-considered quadrature method. When the sample average scheme is applied to the uncertain optimal control problem, it produces a sequence of high-dimensional nonlinear optimal control problems. As in the quadrature approach considered in Chapter 3, these approximate problems can be solved using existing techniques from computational optimal control [29, 41, 79].

The aim of this chapter is the rigorous analysis of the convergence properties of algorithms for uncertain optimal control which are based on this technique of sample average approximation. Because the collocation nodes for the parameter space are selected randomly, the convergence results of the previous chapter can not be directly extended to this case. Instead, we establish convergence properties for the sample average method by leveraging existing results for direct approximation schemes in computational optimal control [41,61,79] and an extension of the strong law of large numbers (see [3,5]). In addition, we establish a necessary condition for optimality for both the

unconstrained and constrained problems in the form of an optimality function based on the $\mathcal{L}_2$-Frechet derivative of the objective functional. We demonstrate the approximate based on a sample average scheme is consistent in the sense of Polak [61, Section 3.3] by analyzing the convergence of the objective and optimality functions. This property guarantees that an accumulation point of a sequence of optimal (stationary) points of the approximate problem will be an optimal (stationary) point to the original problem.

4.1 Formulation of the Uncertain Control Problem

The focus of this chapter is a computational method for the optimal control of uncertain systems using a sample average approach. Because we our convergence analysis for this method utilizes an extension of the strong law of large numbers rather than the quadrature approach of Chapter 3, we require a different problem setting and new regularity assumptions on the functions used in the problem formulation. In this problem formulation both the cost functional and system dynamics may depend on stochastic parameters, and the objective is to find a control which minimizes the expectation of the cost functional over a probability space of possible parameter values. In this problem, the goal is to find is to find an initial state and control pair $\eta = (\xi^\eta, u^\eta)$ that minimizes the objective functional

$$J(\eta) = \mathbb{E}^P[F(x^\eta(1,\omega),\omega)]. \tag{4.1}$$

Here $\mathbb{E}^P$ is the expectation on the probability space (Ω, Σ, P) where the sigma-field Σ is complete with respect to the measure P and $\omega \in \Omega$. Furthermore, $x^\eta(t,\omega)$ is the

solution to the uncertain dynamical system

$$\dot{x}^\eta(t,\omega) = f(x^\eta(t,\omega), u^\eta(t), \omega), \qquad\qquad x(0,\omega) = \xi^\eta + \iota(\omega), \qquad (4.2)$$

almost surely. Here $\xi^\eta \in \mathbb{R}^n$, $u^\eta : [0,1] \mapsto \mathbb{R}^m$, $x : [0,1] \times \Omega \mapsto \mathbb{R}^n$, $\iota : \Omega \mapsto \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \mapsto \mathbb{R}^n$, and $F : \mathbb{R}^n \times \Omega \mapsto \mathbb{R}$. Note that for a fixed $\omega \in \Omega$, (4.2) is a standard deterministic dynamical system, therefore the existence and uniqueness of the solution can be guaranteed under suitable regularity conditions. Such conditions assure that the objective functional (4.1) is well-defined. Note that this problem formulation is more general than the Problem B considered in Chapter 3 in that the parameter space is not required to be a compact Euclidean space, and the initial condition is included as a decision variable.

In this chapter we consider both constrained and unconstrained optimal control problems for the objective functional (4.1) and dynamics (4.2). Before we define these problems, we introduce the spaces on which we conduct our analysis. To develop optimality conditions, we make use of an inner product on the space of decision variables. Therefore we work in the $\mathcal{L}_2$ topology. Let $\mathcal{L}_2^m[0,1]$ be the space of all functions $v : [0,1] \mapsto \mathbb{R}^m$ such that $\int_0^1 \|v(t)\|^2 \, dt < \infty$. We carry out our analysis in a subspace of the Hilbert space

$$H_2 = \mathbb{R}^n \times \mathcal{L}_2^m[0,1],$$

where the inner product and norm on H_2 are defined for any $\eta = (\xi^\eta, u^\eta), \eta' = (\xi^{\eta'}, u^{\eta'}) \in$

H_2 by

$$\langle \eta, \eta' \rangle_{H_2} = \langle \xi^\eta, \xi^{\eta'} \rangle + \langle u^\eta, u^{\eta'} \rangle_2.$$

Therefore the norm in H_2 is given by

$$\|\eta\|_{H_2}^2 = \|\xi^\eta\|^2 + \|u^\eta\|_2^2.$$

In this paper we address the two cases of the uncertain optimal control problem, where the control $u(t)$ is constrained to be in either a compact convex set or an open convex set in $\mathbb{R}^m$ for almost every $t \in [0,1]$. We therefore define the admissable sets for each of these problems as follows: given compact, convex sets $\Xi_C \subset \mathbb{R}^n$ and $U_C \subset \mathbb{R}^m$, we define the set of admissable controls

$$\mathbf{U}_C = \{u \in L_2^m[0,1] \,|\, u(t) \in U_C \text{ for almost every } t \in [0,1]\}.$$

The set of all admissable state-control pairs for this problem is then given by $\mathbf{H}_C = \Xi_C \times \mathbf{U}_C$. Similarly given bounded, open, convex sets $\Xi_O \subset \mathbb{R}^n$ and $U_O \subset \mathbb{R}$, we define the set of admissable controls

$$\mathbf{U}_O = \{u \in L_2^m[0,1] \,|\, u(t) \in U_O \text{ for almost every } t \in [0,1]\}.$$

The set of all admissable state-control pairs for this problem is then given by $\mathbf{H}_O = \Xi_O \times \mathbf{U}_O$.

The sets $\mathbf{H}_C$ and $\mathbf{H}_O$ are a subsets of the pre-Hilbert space $H_{\infty,2} = \{(\xi, u) \in H_2 | \|u\|_\infty < \infty\}$. For mathematical convenience, we assume $\Xi_O \subset \Xi_C$ and $U_O \subset U_C$ so that $\mathbf{H}_O \subset \mathbf{H}_C$. We observe that in this work we define the admissable set differently than in Polak [61, Chapter 4], which requires the pointwise control constraint be satisfied

for all $t \in [0,1]$. Let $U \subset \mathbb{R}^m$. We note that for each $u \in \mathcal{L}_2^m[0,1]$ with $u(t) \in U$ for almost every $t \in [0,1]$, there is a member $\tilde{u}$ of its equivalence class such that $\tilde{u}(t) \in U$ for every $t \in [0,1]$. Therefore for any given constraint set U, we can apply the standard results from the theory of differential equations to controls from our admissable set.

In developing optimality conditions, we evaluate derivatives with respect to the decision variable η. In order to guarantee that these derivatives exist, we work on a space $\mathbf{H}$ which is slightly larger than $\mathbf{H}_C$. To define the space $\mathbf{H}$, let $\rho_1, \rho_2 \in \mathbb{R}$ be constants large enough so that $\|\xi^\eta\| < \rho_1, \|u^\eta\|_\infty < \rho_2$, for all $\eta \in \mathbf{H}_C$. The existence of these constants is guaranteed by the compactness of Ξ_C and U_C. Now let $\mathbf{H} = \{(\xi, u) \in \mathbb{R}^n \times \mathcal{L}_2^m[0,1] \mid \|\xi\| < \rho_1, \|u\|_\infty < \rho_2\}$. The space $\mathbf{H}$ is open in the $\mathcal{L}_\infty$ topology and the inclusion $\mathbf{H}_O \subset \mathbf{H}_C \subset \mathbf{H}$ holds. The reader should note that all convergence results on the sets $\mathbf{H}_O, \mathbf{H}_C$, and $\mathbf{H}$ are with respect to the $\mathcal{L}_2$ topology.

With the appropriate function spaces defined, we now state the uncertain optimal control problems that are the focus of this work:

Problem $\mathbf{D_C}$: Find an initial state and control pair $\eta = (\xi^\eta, u^\eta) \in \mathbf{H}_C$ to minimize the objective functional (4.1) subject to the uncertain dynamical system (4.2).

Problem $\mathbf{D_O}$: Find an initial state and control pair $\eta = (\xi^\eta, u^\eta) \in \mathbf{H}_O$ to minimize the objective functional (4.1) subject to the uncertain dynamical system (4.2).

Because the constraint set U_C for Problem D_C is closed, this formulation can be used to approach uncertain optimal control problems with inequality constraints, as long as set of points which satisfy these constraints is compact and convex. Problem D_O can be used to approach unconstrained optimal control problems by making U_O

large enough that all reasonable controls lie in the admissable set. To conduct analysis of Problems D_C and D_O we need the following regularity assumptions:

Assumption 10. *There exists a compact set $X_0 \subset \mathbb{R}^m$ such that for each $\eta \in \mathbf{H}$, $x^\eta(t, \omega) \in X_0$ for all $t \in [0, 1], \omega \in \Omega$, where x^η is the solution to (4.2) for $\eta = (\xi^\eta, u^\eta)$.*

This assumption essentially requires that there does not exist $\omega \in \Omega$ such that the dynamical system given by $f(\cdot, \cdot, \omega)$ has a finite escape time. This assumption will be valid for a number of dynamical systems frequently encountered in control problems, for example input-to-state stable systems and systems for which f is globally Lipschitz or satisfies a linear growth condition in the state variable.

Assumption 11. *For the set X_0 defined in Assumption 10 and the set $V = \{v \in \mathbb{R}^m | \|v\| < \rho_2\}$, for each $\omega \in \Omega$ the function $f(\cdot, \cdot, \omega)$ is continuously differentiable on $X_0 \times V$ and for each $x \in X_0, v \in V, f(x, v, \cdot)$ is measurable and bounded on Ω. Furthermore, there exists a measurable function $L_f : \Omega \mapsto [1, \infty)$ such that for all $x', x'' \in X_0$, and $v', v'' \in V$, the following inequalities hold for every $\omega \in \Omega$:*

$$\left\| f(x', v', \omega) - f(x'', v'', \omega) \right\| \leq L_f(\omega) \left[\left\| x' - x'' \right\| + \left\| v' - v'' \right\| \right],$$

$$\left\| f_x(x', v', \omega) - f_x(x'', v'', \omega) \right\| \leq L_f(\omega) \left[\left\| x' - x'' \right\| + \left\| v' - v'' \right\| \right],$$

$$\left\| f_u(x', v', \omega) - f_u(x'', v'', \omega) \right\| \leq L_f(\omega) \left[\left\| x' - x'' \right\| + \left\| v' - v'' \right\| \right].$$

Assumption 12. *For the set X_0 defined in Assumption 10, $F(\cdot, \omega)$ is continuously differentiable on X_0 for each $\omega \in \Omega$, and $F(x, \cdot), F_x(x, \cdot)$ are measurable for each $x \in X_0$. Furthermore, there exists a measurable function $L_F : \Omega \mapsto [1, \infty)$ such that for any*

$x', x'' \in X_0$, the following inequalities hold for every $\omega \in \Omega$:

$$\left\| F(x', \omega) - F(x'', \omega) \right\| \leq L_F(\omega) \left\| x' - x'' \right\|, \quad \left\| F_x(x', \omega) - F_x(x'', \omega) \right\| \leq L_F(\omega) \left\| x' - x'' \right\|.$$

Assumptions 11-12 require the differentiability of the functions in the problem formulation with respect to the states and controls, as well as measurability and integrability of the Lipschitz constant with respect to the stochastic parameter ω. These assumptions will be valid for a variety of problem frameworks in physical and other applications. For instance, in the optimal search and ensemble control settings introduced in Sections 1.1 and 1.2, the parameter space is a compact subspace of $\mathbb{R}^n$ and the functions in the problem formulation are sufficiently smooth, therefore Assumption 12 is valid in these cases. These assumptions are used later to establish convergence properties and optimality conditions for Problems D_C and D_O.

In order to facilitate the analysis of the computational framework for Problems D_C and D_O, we first state the following results on uncertain dynamical systems.

Proposition 4.1.1. *Suppose that Assumptions 10-11 are satisfied. Then, for any $\eta \in$* **H***, the uncertain dynamical systems (4.2) has a unique solution $x^\eta(\cdot, \omega)$ for each $\omega \in \Omega$.*

Proof. Follows directly from Proposition 5.6.5 of [61]. $\qquad\square$

Proposition 4.1.2. *[2, Lemma 4.51](Carathéodory Functions are Jointly Measurable) Let (S, Σ) be a measurable space, X a separable metric space, and Y a metrizable space. Let $f : X \times S \mapsto Y$ be a function such that*

i) for each $x \in X$, $f(x, \cdot) : S \mapsto Y$ is measurable;

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ii) for each $s \in S, f(\cdot, s) : X \mapsto Y$ is continuous.

Then f *is called a Caratheodory function and* $f : X \times S \mapsto Y$ *is jointly measurable.*

Lemma 4.1.3. *Suppose that Assumption 10 is satisfied, and let V be the set defined in Assumption 11. Let $\kappa : \mathbb{R}^l \times V \times \Omega \mapsto \mathbb{R}^l$ be such that $\kappa(\cdot, \cdot, \omega)$ is continuously differentiable for each $\omega \in \Omega$ and $\kappa(x, u, \cdot)$ is measurable for each $x \in \mathbb{R}^l, v \in V$. Suppose also that there exists a measurable function $K : \Omega \mapsto [1, \infty)$ such that for every $x, x' \in \mathbb{R}^n$, and $v, v' \in V$, and $\omega \in \Omega$,*

$$\left\| \kappa(x, v, \omega) - \kappa(x', v', \omega) \right\| \leq K(\omega) \left[\left\| x - x' \right\| + \left\| v - v' \right\| \right].$$

For each $\eta = (\xi^\eta, u^\eta) \in \mathbf{H}, \omega \in \Omega$, let $\chi^\eta : [0, 1] \times \Omega \to \mathbb{R}^l$ be the solution to

$$\dot{\chi}^\eta(t, \omega) = \kappa(\chi^\eta(t, \omega), u^\eta(t), \omega), \qquad \chi(0) = \xi^\eta.$$

Then χ^η is measurable and for each $\omega \in \Omega$ we have

$$\left\| x^{\eta'}(t, \omega) - x^{\eta''}(t, \omega) \right\| \leq \sqrt{2} K(\omega) e^{K(\omega)} \left\| \eta' - \eta'' \right\|_{H_2}.$$

Proof. Let $\eta = (\xi^\eta, u^\eta) \in \mathbf{H}$. Let $\chi_0^\eta : [0, 1] \times \Omega \mapsto \mathbb{R}^l$ be such that $\chi_0^\eta(0, \omega) = \xi^\eta$ for each $\omega \in \Omega$, $\chi^\eta(\cdot, \omega)$ is absolutely continuous, and $\chi_0^\eta(t, \cdot)$ is measurable. Then we define a sequence of functions $\{\chi_n^\eta\}_{n=0}^\infty$ satisfying

$$\chi_{n+1}^\eta(t, \omega) = \xi^\eta + \int_0^t \kappa(\chi_n^\eta(s, \omega), u^\eta(s), \omega) ds.$$

We demonstrate the measurability of χ_n^η, for each $n \in \mathbb{N}$ by induction. For a given $n \in \mathbb{N}, t \in [0, 1]$, consider the function $\psi_n : [0, t] \times \Omega \to \mathbb{R}^l$,

$$\psi_n(s, \omega) = \kappa(\chi_n^\eta(s, \omega), u(s), \omega).$$

For each $n \in \mathbb{N}$, if χ_n^η is measurable, then ψ_n is a Carathéodory function and therefore measurable by Proposition 4.1.2, and thus $\chi_{n+1}^\eta(t, \cdot) = \xi^\eta + \int_0^t \psi_n(s, \cdot)ds$ is measurable. The function χ_0 is Carathéodory by definition and therefore measurable, which implies that χ_n^η is measurable for each $n \in \mathbb{N}$ by induction. By the proof of Picard's Lemma (see [61, Lemma 5.6.3]), we have $\chi_n^\eta(\cdot, \omega) \to \chi^\eta(\cdot, \omega)$ pointwise for each $\omega \in \Omega$. The function χ^η is thus a pointwise limit of measurable functions and is therefore measurable. It follows from the proof of Lemma 5.6.7 of [61] that for $\tilde{L}(\omega) = \sqrt{2}K(\omega)e^{K(\omega)}$, for all $\eta', \eta'' \in \mathbf{H}, \omega \in \Omega$, and $t \in [0, 1]$,

$$\left\| \chi^{\eta'}(t, \omega) - \chi^{\eta''}(t, \omega) \right\| \leq \tilde{L} \left\| \eta' - \eta'' \right\|_{H_2},$$

and the conclusion follows. $\square$

4.2 Approximation Using a Sample Average Scheme

In this section we introduce the approximate optimal control problem based on a sample average scheme. Sample average approximations have been successfully applied to a wide variety of problems from the field of stochastic optimization with finite-dimensional decision spaces [81]. In the sample average approach, a random sample of parameter values is drawn from the parameter space, and the expectation in the objective functional is approximated by the sample mean. When the sample average approximation is applied to a stochastic programming problem with a finite-dimensional decision space, this process results in a sequence of approximating nonlinear programming problems. In this work we use the sample average method to approximate Prob-

lems D_C and D_O, which have an infinite-dimensional decision space. The resulting approximate problem is a standard optimal control problem that can be solved using existing techniques from the field of control theory [12]. In addition, we use an extension of the Strong Law of Large Numbers (see [3,5]) to analyze the convergence properties of such an approximation.

To apply this approximation scheme, for a given sample size M, we take an independent P-distributed sample $\{\omega_1, \omega_2, \cdots, \omega_M\}$ from the parameter space Ω and approximate the objective functional (4.1) by the sample average

$$J^M(\eta) \doteq \frac{1}{M} \sum_{i=1}^{M} F\big(x^\eta(1, \omega_i), \omega_i\big). \qquad (4.3)$$

The approximate uncertain optimal control problems can then be stated as follows:

Problem $\mathbf{D_C^M}$: find $\eta \in \mathbf{H}_C$ to minimize the objective functional (4.3), where x^η is the solution to the uncertain dynamical system (4.2).

Problem $\mathbf{D_O^M}$: find $\eta \in \mathbf{H}_O$ to minimize the objective functional (4.3), where x^η is the solution to the uncertain dynamical system (4.2).

We discuss the convergence properties of Problems D_C^M and D_O^M in the context of epiconvergence of the objective functionals. The concept of epiconvergence provides a natural framework to analyze the approximation of an optimization problem, as it allows us to discuss the convergence of the inf and arg min operators. For a survey of preliminary results on epiconvergence and stochastic optimization, see Section 2.4. To demonstrate the epiconvergence of the approximate objective functional J^M to the original objective functional J, we show that J can be written as the expectation of a

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random lower semi-continuous function. To this end we introduce $T : \mathbf{H} \times \Omega$ given by

$$T(\eta, \omega) \doteq F(x^{\eta}(1, \omega), \omega)$$

The following lemma establishes that T is a random lower semi-continuous function.

Lemma 4.2.1. *For each $\omega \in \Omega$, the function $T(\cdot, \omega)$ is Lipschitz continuous with Lipschitz constant $L_T(\omega) = \sqrt{2} L_F(\omega) L_f(\omega) e^{L_f(\omega)}$. Furthermore, T is $\mathcal{B} \otimes \Sigma$ measurable, where $\mathcal{B}$ is the Borel sigma-field generated by the open sets of $\mathbf{H}$.*

Proof. From Assumption 11 and Lemma 5.6.7 of Ref. [61], it is known that for each $\eta, \eta' \in \mathbf{H}$ and $\omega \in \Omega$,

$$\left\| x^{\eta}(1, \omega) - x^{\eta'}(1, \omega) \right\| \leq \sqrt{2} L_f(\omega) e^{L_f(\omega)} \left\| \eta - \eta' \right\|_{H_2}.$$

It follows from Assumption 12 that

$$\left| T(\eta, \omega) - T(\eta', \omega) \right| \leq \sqrt{2} L_F(\omega) L_f(\omega) e^{L_f(\omega)} \left\| \eta - \eta' \right\|_{H_2}.$$

$F : \mathbb{R}^n \times \Omega \mapsto \mathbb{R}$ is measurable by Assumption 12 and Proposition 4.1.2. For each $\eta \in \mathbf{H}$, $x^{\eta}(1, \cdot)$ is measurable by Lemma 4.1.3, so that $T(\eta, \cdot) = F(x^{\eta}(1, \cdot), \cdot)$ is measurable. T is therefore $\mathcal{B} \times \Sigma$ measurable by Proposition 4.1.2. $\qquad \square$

We can now write the objective functional J and approximate objective functional J^M in terms of the function random lower semi-continuous function T:

$$J(\eta) = \mathbb{E}^P \left[T(\eta, \omega) \right], \qquad\qquad J^M(\eta) = \frac{1}{M} \sum_{i=1}^{M} T(\eta, \omega_i).$$

Before we can establish the epiconvergence $J^M \to^{\mathrm{epi}} J$ using Proposition 2.4.1, we must show that the decision space is a complete, separable metric space.

Lemma 4.2.2. *The space* $\mathbf{H}_C$ *is a complete, separable metric space.*

Proof. As a subset of the separable metric space H_2, $\mathbf{H}_C$ is separable. To establish completeness, we show that $\mathbf{H}_C$ is a closed subset of the complete space H_2. Suppose that there is a sequence $u_k \in \mathbf{U}_C$, but $u_k \to u \in \mathcal{L}_2^m[0,1]$ with $u \notin \mathbf{U}_C$. Define $d_U : \mathbb{R}^m \mapsto \mathbb{R}$ by $d_U(v) = \min_{\nu \in U_C} \|\nu - v\|$. Because U_C is compact, d_U is well defined. Now let $A = \{t \in [0,1] | u(t) \notin U_C\}$ and $A_j = \{t \in [0,1] | d_U(u(t)) > \frac{1}{j}\}$. Note that $\mu(A) > 0$, where μ is the Lebesgue measure on $[0,1]$. Since U_C is closed, if $v \in \mathbb{R}^m$ but $v \notin U_C$, then $d_U(v) > 0$. Therefore $A = \{t \in [0,1] | d_U(u(t)) > 0\}$. Because $A = \cup_{j \in \mathbb{N}} A_j$ and $\mu(A) > 0$, there must exist $j \in \mathbb{N}$ such that $m(A_j) > 0$. Then:

$$\|u_k - u\|_2 = \left(\int_0^1 \|u_k(t) - u(t)\|^2 \, dt\right)^{\frac{1}{2}} \geq \left(\int_{A_j} \|u_k(t) - u(t)\|^2 \, dt\right)^{\frac{1}{2}}$$

$$\geq \left(\int_{A_j} [d_U(u(t))]^2 \, dt\right)^{\frac{1}{2}}$$

$$\geq \frac{1}{j}\sqrt{\mu(A_j)}.$$

This is a contradiction, therefore $\mathbf{U}_C$ is closed in $\mathcal{L}_2^m[0,1]$. As a closed subset of a complete space, $\mathbf{U}_C$ is complete, and therefore $\mathbf{H}_C = \Xi_C \times \mathbf{U}_C$ is complete. $\qquad\square$

We can now demonstrate the epiconvergence of the approximate objective functional using the following assumption.

Assumption 13. *Let* $L_T : \Omega \mapsto [1, \infty)$ *be defined as in Lemma 4.2.1. Then* $L_T \in \mathcal{L}^1(\Omega)$.

Note that this assumption is valid when Ω is a compact subset of $\mathbb{R}^d$ and the functions f and F are continuously differentiable with respect to ω.

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Theorem 4.2.3. *Suppose that Assumptions 10-13 hold. Then J^M epiconverges almost surely to J on $\mathbf{H}_C$ and J^M epiconverges almost surely to J on $\mathbf{H}_O$ as $M \to \infty$.*

Proof. By Lemma 4.2.2, $\mathbf{H}_C$ is a separable complete metric space. By Lemma 4.2.1, T is $\mathcal{B} \otimes \Sigma$ measurable and there exists scalars a and b such that $T(\eta, \omega) \geq a + bL_T(\omega)$ for all $\eta \in \mathbf{H}_C$. By Assumption 13 the function $a + bL_T(\omega)$ is integrable. The convergence $J^M\big|_{\mathbf{H}_C} \to^{\mathrm{epi}} J\big|_{\mathbf{H}_C}$ almost surely then follows from Theorem 2.4.1. This convergence, together with the fact that $J^M(\eta) \to J(\eta)$ almost surely for all $\eta \in \mathbf{H}_O$ establishes the convergence $J^M\big|_{\mathbf{H}_O} \to^{\mathrm{epi}} J\big|_{\mathbf{H}_O}$ almost surely. $\qquad\square$

Theorem 4.2.3 and Proposition 2.3.1 show that our sample average scheme has the property that accumulation points of a sequence of global minimizers to the approximate problem are global minimizers of the original problem.

4.3   Optimality Conditions

Absent convexity, it is not generally possible to determine whether a numerically computed solution to an optimal control problem is a global minimizer. Necessary conditions, such as Pontryagin's Minimum Principle [33, 63], provide a method to assess the optimality of a numerically computed solution. Polak [61, Chapter 4], provides necessary conditions for the standard nonlinear optimal control problem in terms of optimality functions, which determine the stationary points of the objective functional. In this section we apply this approach to derive optimality functions for the non-standard uncertain Problems D_C, D_O, D_C^M, and D_O^M which are based on the $\mathcal{L}_2$-Frechet deriva-

tive of the objective functional. To establish the Frechet derivatives of the objective functionals, we first state the Frechet derivative of T.

Proposition 4.3.1. *Suppose that Assumptions 10-12 are satisfied.*

i)   For any $\omega \in \Omega, \eta \in \mathbf{H}$ and $\delta\eta \in H_{\infty,2}, T(\cdot, \omega)$ has a Frechet derivative $DT(\eta; \delta\eta; \omega)$ at η given by $\langle \nabla_\eta T(\eta, \omega), \delta\eta \rangle_{H_2}$

The gradient $\nabla_\eta T(\eta, \omega) = (\nabla_\xi T(\eta, \omega), \nabla_u T(\eta, \omega))^T \in H_{\infty,2}$ is given by

$$\nabla_\xi T(\eta, \omega) = p^\eta(0, \omega), \tag{4.4}$$

$$\nabla_u T(\eta, \omega)(s) = f_u^T(x^\eta(s, \omega), u^\eta(s), \omega)p^\eta(s, \omega), \tag{4.5}$$

and $p^\eta(s, \omega)$ is the solution to the adjoint equation

$$\dot{p}^\eta(s, \omega) = - f_x^T(x^\eta(s, \omega), u(s), \omega)p^\eta(s, \omega) \quad \text{for } s \in [0, 1], \quad p^\eta(1, \omega) = F_x(x^\eta(1, \omega), \omega).$$

$$\tag{4.6}$$

ii)   The gradient $\nabla_\eta T(\cdot, \omega)$ is Lipschitz continuous on $\mathbf{H}_C$.

iii)   For any $\eta \in \mathbf{H}$ and $\delta\eta \in H_{\infty,2}$, $T(\cdot, \omega)$ has a Frechet differential $DT(\eta; \delta\eta; \omega)$ at η.

Proof. The proposition follows directly from Corollary 5.6.9 of [61]. □

The existence of the Frechet derivative in Proposition 4.3.1 allows us to introduce the Frechet derivatives of J and J^M by employing Fubini's theorem.

Lemma 4.3.2. *Suppose that Assumptions 10-13 are satisfied. Then for any $\eta \in \mathbf{H}, \delta\eta \in H_{\infty,2}$:*

i) *J has a Frechet differential $DJ(\eta; \delta\eta)$ at η given by $DJ(\eta; \delta\eta) = \langle \nabla J(\eta), \delta\eta \rangle_{H_2}$*

with the gradient given by

$$\nabla J(\eta) = \mathbb{E}^P \left[\nabla_\eta T(\eta, \omega) \right], \tag{4.7}$$

ii) *The gradient ∇J is Lipschitz continuous on $\mathbf{H}_C$.*

iii) *J^M has a Gateuax differential $DJ^M(\eta; \delta\eta)$ at η given by $DJ^M(\eta; \delta\eta) = \langle \nabla J^M(\eta), \delta\eta \rangle_{H_2}$*

with the gradient given by

$$\nabla J^M(\eta) = \frac{1}{M} \sum_{i=1}^{M} \nabla_\eta T(\eta, \omega_i), \tag{4.8}$$

iv) *The gradient ∇J^M is Lipschitz continuous on $\mathbf{H}_C$.*

Proof. We prove i) and ii); iii) and iv) follow by an identical argument with Ω replaced

by $\{\omega_1, \ldots, \omega_M\}$ and P replaced by the counting measure normalized to 1.

Proof of i): Let $\delta\eta \in H_{\infty,2}, \eta \in \mathbf{H}, \omega \in \Omega$. Because $\mathbf{H}$ is open in the L_∞

topology there exists a $\lambda^* > 0$ such that $\eta + \lambda\delta\eta \in \mathbf{H}$ for all $\lambda \in [0, \lambda^*]$. From Lemma

4.2.1, $T(\cdot, \omega)$ is Lipschitz continuous in η with Lipschitz constant $L_T(\omega)$ for each $\omega \in \Omega$,

and by Assumption we have 13 $L_T(\omega) \in \mathcal{L}^1(\Omega)$. From this fact we have

$$|T(\eta + \lambda\delta\eta, \omega) - T(\eta, \omega)| \leq \left(L_T(\omega) \|\delta\eta\|_{H_2} \right) \lambda.$$

Therefore for each $\omega \in \Omega, \eta \in \mathbf{H}, \lambda \in [0, \lambda^*]$,

$$\left| \frac{T(\eta + \lambda\delta\eta, \omega) - T(\eta, \omega)}{\lambda} \right| \leq L_T(\omega) \|\delta\eta\|_{H_2}.$$

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Then the Gateaux derivative of J is given by:

$$DJ(\eta; \delta\eta) = \lim_{\lambda\downarrow 0} \frac{\mathbb{E}^P\left[T(\eta + \lambda\delta\eta, \omega)\right] - \mathbb{E}^P\left[T(\eta, \omega)\right]}{\lambda}$$

$$= \lim_{\lambda\downarrow 0} \mathbb{E}^P\left[\frac{T(\eta + \lambda\delta\eta, \omega) - T(\eta, \omega)}{\lambda}\right]$$

$$= \mathbb{E}^P\left[\lim_{\lambda\downarrow 0} \frac{T(\eta + \lambda\delta\eta, \omega) - T(\eta, \omega)}{\lambda}\right]$$

$$= \mathbb{E}^P\left[DT(\eta, \delta\eta; \omega)\right],$$

where we have used the dominated convergence theorem. Let $\delta\eta = (\xi^{\delta\eta}, u^{\delta\eta})$. Note that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^1 \left\langle \left[\nabla_u T(\eta, \omega)(t)\right], u^{\delta\eta}(t)\right\rangle dt\right] \leq \mathbb{E}^{\mathbb{P}}\left[\|\nabla T(\eta, \omega)\|_{H_2} \|\delta\eta\|_{H_2}\right]$$

is bounded, so that we can write

$$DJ(\eta; \delta\eta) = \mathbb{E}^P\left[\left\langle \nabla_\xi T(\eta, \omega), \xi^{\delta\eta}\right\rangle\right] + \mathbb{E}^P\left[\int_0^1 \left\langle \nabla_u T(\eta, \omega)(t), u^{\delta\eta}(t)\right\rangle dt\right]$$

$$= \mathbb{E}^P\left[\left\langle \nabla_\xi T(\eta, \omega), \xi^{\delta\eta}\right\rangle\right] + \int_0^1 \mathbb{E}^P\left[\left\langle \nabla_u T(\eta, \omega)(t), u^{\delta\eta}(t)\right\rangle\right] dt$$

$$= \left\langle \mathbb{E}^P\left[\nabla_\xi T(\eta, \omega)\right], \xi^{\delta\eta}\right\rangle + \int_0^1 \left\langle \mathbb{E}^P\left[\nabla_u T(\eta, \omega)(t)\right], u^{\delta\eta}(t)\right\rangle dt$$

$$= \left\langle \mathbb{E}^P\left[\nabla_\eta T(\eta, \omega)\right], \delta\eta\right\rangle_{H_2},$$

where we have used Fubini's theorem. To demonstrate that the Gateaux derivative DJ

is the Frechet derivative of J, consider the quantity

$$\lim_{\|\delta\eta\|_{H_2}\to 0} \frac{\|J(\eta+\delta\eta) - J(\eta) - DJ(\eta;\delta\eta)\|_{H_2}}{\|\delta\eta\|_{H_2}}$$

$$= \lim_{\|\delta\eta\|_{H_2}\to 0} \frac{\left\|\mathbb{E}^P\left[T(\eta+\delta\eta,\omega) - T(\eta,\omega) - DT(\eta;\delta\eta;\omega)\right]\right\|_{H_2}}{\|\delta\eta\|_{H_2}}$$

$$\leq \lim_{\|\delta\eta\|_{H_2}\to 0} \mathbb{E}^P\left[\frac{\|T(\eta+\delta\eta,\omega) - T(\eta,\omega) - DT(\eta;\delta\eta;\omega)\|_{H_2}}{\|\delta\eta\|_{H_2}}\right]$$

$$= \mathbb{E}^P\left[\lim_{\|\delta\eta\|_{H_2}\to 0} \frac{\|T(\eta+\delta\eta,\omega) - T(\eta,\omega) - DT(\eta;\delta\eta;\omega)\|_{H_2}}{\|\delta\eta\|_{H_2}}\right]$$

$$= 0,$$

where we have used dominated convergence.

The proof of ii) follows directly from the Lipschitz continuity of $\nabla_\eta T(\eta,\omega)$. $\qquad\square$

We now introduce non-positive optimality functions for Problem D_C, D_C^M, D_O, and D_O^M, based on the Frechet derivatives defined in Lemma 4.3.2.

$$\theta_C(\eta) = \min_{\eta'\in\mathbf{H}_C} DJ(\eta;\eta'-\eta) + \frac{1}{2}\left\|\eta'-\eta\right\|_{H_2}^2, \tag{4.9}$$

$$\theta_C^M(\eta) = \min_{\eta'\in\mathbf{H}_C} DJ^M(\eta;\eta'-\eta) + \frac{1}{2}\left\|\eta'-\eta\right\|_{H_2}^2, \tag{4.10}$$

$$\theta_O(\eta) = -\frac{1}{2}\left\|\nabla J(\eta)\right\|_{H_2}^2, \tag{4.11}$$

$$\theta_O^M(\eta) = -\frac{1}{2}\left\|\nabla J^M(\eta)\right\|_{H_2}^2. \tag{4.12}$$

Proposition 4.3.3. *Suppose that Assumptions 10-13 hold. Then the following are true:*

i) θ_C is a continuous optimality function for D_C.

ii) θ_C^M is a continuous optimality function for D_C^M.

iii) θ_O is a continuous optimality function for D_O.

iv) θ_O^M *is a continuous optimality function for* D_O^M.

Proof. The proof of $i) - ii)$ follows directly from Lemma 4.3.2 and the arguments used in the proof of Theorem 4.2.3a in [61], with J or J^M replacing f^0, H_C replacing $\mathbf{H}$, Lemma 4.3.2 replacing Corollary 5.6.9. The proof of $iii) - iv)$ follows directly from Lemma 4.3.2 and the arguments used in the proof of Theorem 4.2.3c in [61], with J or J^M replacing f^0, H replacing H^O, Lemma 4.3.2 replacing Corollary 5.6.9. $\qquad\square$

In general, the necessary condition based on the $\mathcal{L}_2$ variation of the objective functional will not be equivalent to the Pontryagin Minimum Principle except in the case where the Hamiltonian is convex in u. However it can be shown that for the Problem D_O, under certain regularity conditions the necessary condition $\theta_O(\eta) = \|\nabla J(\eta)\|_{H_2}^2 = 0$ is equivalent to the stationarity of the Hamiltonian given by

$$\mathcal{H}(x, \lambda, u, t) = \mathbb{E}^P \big[f(x(t, \omega), u(t), \omega)^T p(t, \omega) \big],$$

where p is the adjoint to the state variable x. To see this, suppose the initial condition is fixed, and note that the stationarity of the Hamiltonian implies that

$$\frac{\partial}{\partial u} \mathcal{H}(x, \lambda, u, t) = \mathbb{E}^P \big[f_u(x(t, \omega), u(t), \omega)^T p(t, \omega) \big] = 0$$

for almost all t when f is sufficiently smooth. Therefore

$$\|\nabla J(\eta)\|_{H_2}^2 = \int_0^1 \big\| \mathbb{E}^P \big[f_u(x(t), u(t), \omega) p(t, \omega) \big] \big\|^2 \, dt = 0$$

.

For the Problem D_C, the approach of Ref. [57,58] can be extended to produce a Pontryagin-like necessary condition for global minimizers that are accumulation points

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of global minimizers of the approximate Problem D_C^M. A direct extension of Pontryagin's minimum principle to the UOCP is desirable, as it may lead to insights into new optimization algorithms, but this approach is not pursued here. For work relating to this topic see Ref. [24, pp. 80-82].

4.4 Consistency of the Approximation Using Sample Averages

In Section 4.2 we analyzed the convergence of the approximation scheme for Problems D_C and D_O using the concept of epiconvergence. Epiconvergence of the objective functionals guarantees that accumulation points of a sequence of local minimizers to the approximate problem will be a local minimizer of the original problem. However, epiconvergence is not sufficient to guarantee that accumulation points of a sequence of stationary points to the approximate problem are stationary. In this section, we demonstrate such a property, thus showing that the approximation scheme based on sample averages is consistent in the sense of Polak [61, Section 3.3].

Definition 6. *[61] Let X be a complete separable metric space, let $G^M : X \mapsto \mathbb{R}, G : X \mapsto \mathbb{R}$ be lower semi-continuous functions, and let $\Gamma^M : X \mapsto \mathbb{R}, \Gamma : X \mapsto \mathbb{R}$ be non-positive upper semi-continuous functions. We say that the pair $\{G^M, \Gamma^M\}_{M \in \mathbb{N}}$ is a consistent approximation to the pair $\{G, \Gamma\}$ if:*

i) G_M epiconverges to J.

ii) If $\{x_M\}_{M=1}^{\infty}$ is a sequence converging to x, then $\limsup_{M \to \infty} \Gamma_M(x_M) \leq \Gamma(x)$.

We have already shown the almost sure epiconvergence of the approximate objective functional J^M to the objective functional J in Theorem 4.2.3. To establish the convergence properties of the optimality function θ_C^M, we introduce the following assumption:

Assumption 14. *There exist constants $L_f', L_F' \in [1, \infty)$ such that $L_f(\omega) \leq L_f'$ and $L_F(\omega) \leq L_F'$ almost surely.*

Note that this assumption will be valid in the case that Ω is a compact subset of $\mathbb{R}^{n_\omega}$ and f, F are continuously differentiable. Therefore the assumption is satisfied for previously considered applications Problem B such as optimal search [57, 58] and ensemble control [73, 74, 76].

The following lemma addresses the measurability and continuity of the gradient of the objective functional.

Lemma 4.4.1. *Suppose that Assumptions 10-14 hold. Let $\eta \in \mathbf{H}$. Then the following are true:*

i) $\nabla_\eta T(\eta, \cdot)(\cdot) : \Omega \times [0, 1] \to \mathbb{R}$ *is measurable.*

ii) *There exists a compact set $U_0 \subset \mathbb{R}^m$ such that $\nabla_u T(\eta, \omega)(t) \in U_0$ for all $\eta \in \mathbf{H}, \omega \in \Omega, t \in [0, 1]$.*

iii) There exists $L_{\nabla T}' \in [1, \infty)$ such that $L_{\nabla T}(\omega) \leq L_{\nabla T}'$ almost surely, where $L_{\nabla T}(\omega)$ is the Lipschitz constant of $\nabla T(\cdot, \omega)$.

iv) For each M, $L_{\nabla J^M} \leq L_{\nabla T}'$ almost surely, where $L_{\nabla J^M}$ is the Lipschitz constant of ∇J^M.

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Proof. Part i) follows directly from (4.4-4.5) and the application of Lemma 4.1.3 to the adjoint system (4.6). Part ii) follows from Lipschitz continuity of f_u (Assumption 11) and p (Lemma 4.1.3) and the boundedness of the set $\mathbf{H}$. Part iii) follows from Assumption 14 and the proof of Lemma 5.6.9b of [61]. Part iv) follows from iii) and the fact that $\nabla J^M(\cdot, \omega) = \frac{1}{M} \nabla T(\cdot, \omega_i^M)$ where $\{\omega_i^M\}_{i=1}^M$ is an independent P-distributed random draw from Ω. $\qquad\square$

To simplify notation, for a given $\eta^* \in \mathbf{H}_C$, we introduce the following functions:

i) $\kappa_{\eta^*}^M : \mathbf{H}_C \mapsto \mathbb{R}; \eta \mapsto \left\langle \nabla J^M(\eta^*), \eta \right\rangle_{H_2}$,

ii) $\kappa_{\eta^*} : \mathbf{H}_C \mapsto \mathbb{R}; \eta \mapsto \left\langle \nabla J(\eta^*), \eta \right\rangle_{H_2}$,

iii) $\mu_{\eta^*}^M : \mathbf{H}_C \mapsto \mathbb{R}; \eta \mapsto \left\langle \nabla J^M(\eta), \eta^* \right\rangle_{H_2}$,

iv) $\mu_{\eta^*} : \mathbf{H}_C \mapsto \mathbb{R}; \eta \mapsto \left\langle \nabla J(\eta), \eta^* \right\rangle_{H_2}$.

Lemma 4.4.2. *Suppose that Assumptions 10-13 are satisfied. Then the following hold:*

i) $\kappa_{\eta^*}^M \to \kappa_{\eta^*}$ uniformly almost surely for each η^* in $\mathbf{H}_C$.

ii) $\mu_{\eta^*}^M \to^{epi} \mu_{\eta^*}$ almost surely for each η^* in H_2.

Proof. Proof of i): For a given $t \in [0,1]$, because the $\nabla_u T(\eta, \omega_i)(t)$, for $i = 1, \ldots, M$ are identically distributed, the strong law of large numbers, (4.7), and (4.8) imply that $\nabla J^M(\eta^*)(t) \to \nabla J(\eta^*)(t)$ almost surely. Therefore $\nabla J^M(\eta^*) \to \nabla J(\eta^*)$ pointwise almost surely as $M \to \infty$. Recall that $\|\eta\|_{H_2} \le \rho_1 + \rho_2$ for all $\eta \in \mathbf{H}_C$. Therefore for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for each $M > K$, we have $\left\| \nabla J^M(\eta^*) - \nabla J(\eta^*) \right\|_{H_2} <$

$\frac{\epsilon}{\rho_1+\rho_2}$ by the dominated convergence theorem. Then

$$\left|\kappa^M_{\eta^*}(\eta) - \kappa_{\eta^*}(\eta)\right| = \left|\langle \nabla J^M(\eta^*) - \nabla J(\eta^*), \eta\rangle_{H_2}\right| \le \left\|\nabla J^M(\eta^*) - \nabla J(\eta^*)\right\|_{H_2} \|\eta\|_{H_2}$$

$$< \frac{\epsilon}{\rho_1 + \rho_2}(\rho_1 + \rho_2) = \epsilon.$$

Proof of ii): First note that by Lemma 4.4.1, $\langle \nabla_\eta T(\eta,\omega), \eta^*\rangle_{H_2}$ is continuous in η and measurable in ω and therefore is a random lower semi-continuous function by Proposition 4.1.2 and Definition 4. Because $\mu_{\eta^*}(\cdot) = \mathbb{E}^P \langle \nabla_\eta T(\cdot,\omega), \eta^*\rangle_{H_2}$ by the proof of Lemma 4.3.2, μ_{η^*} is the expectation of a random lower semi-continuous function and is bounded by Lemma 4.4.1iii. The result then follows from (4.7), (4.8) and Proposition 2.4.1. $\qquad\square$

4.4.1 Consistency of the Approximation the Constrained Problem

Lemma 4.4.2 allows us to establish the almost sure consistent approximation of Problem D_C.

Theorem 4.4.3. *Suppose that Assumptions 10-14 hold. Then the sequence $\{J^M, \theta_C^M\}_{M\in\mathbb{N}}$ is almost surely a consistent approximation to the pair $\{J, \theta_C\}$ on the decision space $\mathbf{H}_C$.*

Proof. The almost sure epiconvergence of $J^M\big|_{\mathbf{H}_C}$ to $J\big|_{\mathbf{H}_C}$ is established in Theorem 4.2.3.

It remains to show that $\limsup_{M\to\infty} \theta_C^M(\eta^M) \le \theta_C(\eta)$ whenever $\eta^M \to \eta$.

Suppose that $\eta^M \in \mathbf{H}_C$ and $\eta^M \to \eta$. First we write

$$
\begin{aligned}
\theta_C^M(\eta^M) &= \min_{\eta' \in \mathbf{H}_C} \left\{ \langle \nabla J^M(\eta^M), \eta' - \eta^M \rangle_{H_2} + \frac{1}{2} \left\| \eta' - \eta^M \right\|_{H_2}^2 \right\} \\
&= \min_{\eta' \in \mathbf{H}_C} \left\{ \langle \nabla J^M(\eta^M), \eta' \rangle_{H_2} + \frac{1}{2} \left\| \eta' - \eta^M \right\|_{H_2}^2 \right\} - \langle \nabla J^M(\eta^M), \eta^M \rangle_{H_2} \\
&= \min_{\eta' \in \mathbf{H}_C} \left\{ \langle \nabla J^M(\eta^M) - \nabla J^M(\eta)), \eta' \rangle_{H_2} + \langle \nabla J^M(\eta), \eta' \rangle_{H_2} + \frac{1}{2} \left\| \eta' - \eta^M \right\|_{H_2}^2 \right\} \\
&\quad - \langle \nabla J^M(\eta^M), \eta^M \rangle_{H_2} \\
&= \min_{\eta' \in \mathbf{H}_C} \left\{ \langle \nabla J^M(\eta^M) - \nabla J^M(\eta)), \eta' \rangle_{H_2} + \kappa_\eta^M(\eta') + \frac{1}{2} \left\| \eta' - \eta^M \right\|_{H_2}^2 \right\} \\
&\quad - \langle \nabla J^M(\eta^M), \eta^M - \eta \rangle_{H_2} - \mu_\eta^M(\eta^M). \tag{4.13}
\end{aligned}
$$

Similarly,

$$
\theta_C(\eta) = \min_{\eta' \in \mathbf{H}_C} \left[\kappa_\eta(\eta') + \frac{1}{2} \left\| \eta' - \eta \right\|_{H_2}^2 \right] - \mu_\eta(\eta). \tag{4.14}
$$

We examine the behavior of $\limsup_{M \to \infty} \theta_C^M(\eta^M)$ by looking at each expression in (4.13).

Note that $\mathbf{H}_C$ is bounded, therefore we have by Lemma 4.4.1iv

$$
\langle \nabla J^M(\eta^M) - \nabla J^M(\eta), \eta' \rangle_{H_2} \le \left\| \nabla J^M(\eta^M) - \nabla J^M(\eta) \right\|_{H_2} \left\| \eta' \right\|_{H_2}
$$

$$
\le L_{\nabla J^M} \left\| \eta^M - \eta \right\|_{H_2} \left\| \eta' \right\|_{H_2} \to 0
$$

uniformly in η' on $\mathbf{H}_C$. Similarly, because

$$
\left\| \eta' - \eta^M \right\|_{H_2}^2 - \left\| \eta' - \eta \right\|_{H_2}^2 = \left\| \eta^M \right\|_{H_2}^2 - \left\| \eta \right\|_{H_2}^2 + 2\langle \eta - \eta^M, \eta' \rangle_{H_2} \to 0
$$

uniformly in η' to 0 on $\mathbf{H}_C$, we have $\|\eta' - \eta^M\|_{H_2}^2 \to \|\eta' - \eta\|_{H_2}^2$ uniformly in η'. This, combined with the uniform convergence $\kappa_\eta^M \to \kappa_\eta$ shows that

$$\min_{\eta' \in \mathbf{H}_C} \left\langle \nabla J^M(\eta^M) - \nabla J^M(\eta)), \eta' \right\rangle_{H_2} + \kappa_\eta^M(\eta') + \frac{1}{2} \|\eta' - \eta^M\|_{H_2}^2$$

$$\to \min_{\eta' \in \mathbf{H}_C} \kappa_\eta(\eta') + \frac{1}{2} \|\eta' - \eta\|_{H_2}^2 \qquad (4.15)$$

Because $\langle \nabla J^M(\eta^M), \eta^M - \eta \rangle_{H_2} \to 0$ almost surely and μ_η^M epiconverges to μ_η, we have, from (4.13-4.15)

$$\limsup_{M \to \infty} \theta_C^M(\eta^M) \le \theta_C(\eta) \text{ almost surely.}$$

$\square$

4.4.2 Consistency of the Approximation of the Unconstrained Problem

We now demonstrate the almost sure consistent approximation of Problem D_O.

Theorem 4.4.4. *Suppose that Assumptions 10-13 hold. Then the sequence $\{J^M, \theta_O^M\}_{M \in \mathbb{N}}$ is almost surely a consistent approximation to the pair $\{J, \theta_O\}$ on the decision space $\mathbf{H}_O$.*

Proof. The almost sure epiconvergence $J^M\big|_{H_O} \to^{\text{epi}} J^M\big|_{H_O}$ was established in Theorem 4.2.3; it remains to establish the convergence properties of the optimality functions. Suppose that $\eta^M \in \mathbf{H}_O$ and $\eta^M \to \eta \in \mathbf{H}_O$. Recall that H_2 is a complete Hilbert space. By Lemma 4.4.2ii and the Riesz Representation theorem, for each $f \in H_2^*$ we have $\liminf_{M \to \infty} f(\nabla J^M(\eta^M)) \ge f(\nabla J(\eta))$ almost surely. By the Hahn-Banach theorem there exists $f^* \in H_2^*$ such that $\|f^*\|_{H_2^*} = 1$ and $f^*(\nabla J(\eta)) = \|\nabla J(\eta)\|_{H_2}$. Furthermore,

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for each M, we have

$$f^*(\nabla J^M(\eta^M)) \leq \|f^*\|_{H_2^*} \left\|\nabla J^M(\eta^M))\right\|_{H_2} = \left\|\nabla J^M(\eta^M))\right\|_{H_2}$$

Therefore

$$\|\nabla J(\eta)\|_{H_2} = f^*(\nabla J(\eta)) \leq \lim_{M \to \infty} \inf f^*(\nabla J^M(\eta^M)) \leq \lim_{M \to \infty} \inf \left\|\nabla J^M(\eta^M)\right\|_{H_2}$$

Therefore $\limsup_{M \to \infty} \theta_O^M(\eta^M) \leq \theta_O(\eta)$ almost surely.

$\square$

4.5   The Time-Discretized Problem

In Sections 4.2-4.4 we analyzed a computational framework for the uncertain optimal control problem based on sample average approximations. This process creates a sequence of approximating standard optimal control problems which can be solved using existing techniques. In this section, we address the convergence properties of the method which solves the approximate Problem D_C^M using the Euler discretization, although this approach can be generalized to other direct discretization algorithms such as Runge-Kutta [41, 79].

First we introduce the framework with which we will perform our discrete approximation, following the framework of Polak [61, Chapter 5]. This will involve an approximation of the admissable set as well as an approximation of the objective

functional. For $k \in \{0, 1, \ldots, N-1\}$, let

$$\pi_{N,k}(t) = \begin{cases} \sqrt{N} & \text{for all } t \in [k/N, (k+1)/N), \text{if } k \leq N-1, \\ \\ 0, & \text{otherwise} \end{cases} \tag{4.16}$$

For any integer $N \geq 1$, we define the subspace $\mathcal{L}_N \subset \mathcal{L}^m_{\infty,2}[0,1]$, by

$$\mathcal{L}_N = \{u \in \mathcal{L}^m_{\infty,2}[0,1] | u(t) = \sum_{k=0}^{N-1} u_k \pi_{N,k}(t)\},$$

and

$$H_N = \mathbb{R}^n \times \mathcal{L}_N \subset H_{\infty,2}.$$

We then define the admissable set for the approximate problem as

$$\mathbf{H}_{C,N} = \mathbf{H}_C \cap H_N.$$

$\mathbf{H}_{C,N}$ is the set of all admissable initial state and control pairs for Problem D_C, with the additional requirement that the control be constant on each interval $[\frac{i}{N}, \frac{i+1}{N})$ for $i \in \{0, \ldots, N-1\}$.

For each $\omega \in \Omega$ and $\eta \in H_N$, we approximate the dynamics (4.2) using the Euler integration formula:

$$x^\eta_N((k+1)/N, \omega) - x^\eta_N(k/N, \omega) = \frac{1}{N} f(x^\eta_N(k/N, \omega), u^\eta(k/N)), k \in \{0, \ldots, N-1\}$$

$$x^\eta_N(0, \omega) = \xi^\eta + \iota(\omega)$$

For a detailed derivation of this approximation scheme and its relation to the nonlinear programming problem, see Polak [61, Chapter 5].

Recall that the objective functional to the Problem D_C is given by $J(\eta) = \mathbb{E}^P[T(\eta, \omega)]$ where $T(\eta, \omega) = F(x^\eta(1, \omega), \omega)$. Let $T^N : \mathbf{H} \times \Omega \mapsto \mathbb{R}$ be the time-discretized approximation to T, i.e. $T^N(\eta, \omega) = F(x_N^\eta(1, \omega), \omega)$. Given a random P-distributed draw $\{\omega_1, \ldots, \omega_M\}$ from Ω, we can define the sample average and time-discretized approximation to the objective functional J by

$$J^{MN} = \frac{1}{M} \sum_{i=1}^{M} T^N(\eta, \omega_i). \tag{4.17}$$

Combining this objective functional with the discretized dynamics

$$x_N^\eta((k+1)/N, \omega_i) - x_N^\eta(k/N, \omega_i) = \frac{1}{N} f(x_N^\eta(k/N, \omega_i), u^\eta(k/N)), \tag{4.18}$$

$$k \in \{0, \ldots, N-1\}, \imath \in \{1, \ldots, M\}$$

$$x_N^\eta(0, \omega_i) = \xi^\eta + \iota(\omega_i), \imath \in \{1, \ldots, M\}, \tag{4.19}$$

we can define the fully discretized problem.

Problem $\mathbf{B_C^{MN}}$: Find an initial state and control pair $\eta = (u^\eta, \xi^\eta) \in \mathbf{H}_{C,N}$ to minimize the objective functional (4.17) subject to the constraints (4.19).

In order to approximate Problem D_C by Problem D_C^{MN}, our desire is to assign to each sample size $M \in \mathbb{N}$ a number $\mathcal{N}(M)$ of time discretization nodes in such a way that $J^{M\mathcal{N}(M)} \to^{\mathrm{epi}} J$. To this end we introduce the following assumption:

Assumption 15. *For the function $\mathcal{N} : \mathbb{N} \mapsto \mathbb{N}$, we have $\mathcal{N}(M) \to \infty$ as $M \to \infty$.*

In Section 4.2 we showed that $J^M \to^{\mathrm{epi}} J$ as $M \to \infty$. It is well known that $J^{MN} \to^{\mathrm{epi}} J^M$ as $N \to \infty$ (see Ref. [61, Chapter 4]). However, these conditions are not sufficient to guarantee that $J^{M\mathcal{N}(M)} \to^{\mathrm{epi}} J$ as $M \to \infty$ for arbitrary assignments

$\mathcal{N} : \mathbb{N} \to \mathbb{N}$. We demonstrate such a property by analyzing the error introduced by the time discretization approximation. Our approach will be based on the fact that the effect of such a time discretization on a standard optimal control problem (which we can consider as a special case in which the value of the parameter ω is fixed) is known and is determined by L_f, L_F. That is, we can use existing results to uniformly bound (in both η and ω) the error introduced to Problem D_C by approximating $T(\eta, \omega)$ by $T^N(\eta, \omega)$.

Proposition 4.5.1. *Suppose that Assumptions 10-15 are satisfied. Then there exists an K_T such that for any $\eta = (\xi, u) \in \mathbf{H}_C$, we have*

$$|T^N(\eta, \omega) - T(\eta, \omega)| \leq K_T/N \tag{4.20}$$

for every $\omega \in \Omega$.

Proof. Follows from Assumption 14, the boundedness of the set $\mathbf{H}_C$, and the proofs of Theorems 5.6.23 and 5.6.24 in Ref. [61]. □

The fact that this convergence is uniform in both η and ω allows us to address the convergence $J^{MN} \to^{\text{epi}} J$.

Theorem 4.5.2. *Suppose Assumptions 10-15 are satisfied. Then $J^{M\mathcal{N}(M)} \to^{\text{epi}} J$ almost surely.*

Proof. In order to establish epiconvergence we must show that

i) $\liminf J^{M\mathcal{N}(M)}(\eta_M) \geq J(\eta)$ whenever $\eta_M \to \eta$,

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ii) $\lim J^{M\mathcal{N}(M)}(\eta_M) = J(\eta)$ for at least one sequence $\eta_M \to \eta$.

To do so, note that Assumption 14 implies the existence of a constant $L_T \in [1, \infty)$ such that $|T(\eta, \omega) - T(\eta', \omega)| \le L_T \|\eta - \eta'\|$ for all $\eta, \eta' \in \mathbf{H}_C, \omega \in \Omega$. Then consider the difference

$$
\begin{aligned}
\left| J(\eta) - J^{M\mathcal{N}(M)}(\eta_M) \right| &\le \left| J(\eta) - J^M(\eta_M) \right| + \left| J^M(\eta_M) - J^{M\mathcal{N}(M)}(\eta_M) \right| \\
&= \left| J(\eta) - J^M(\eta_M) \right| + \left| \frac{1}{M} \sum_{i=1}^{M} T(\eta, \omega_i) - T^{\mathcal{N}(M)}(\eta_M, \omega_i) \right| \\
&\le \left| J(\eta) - J^M(\eta_M) \right| + \left| \frac{1}{M} \sum_{i=1}^{M} T(\eta, \omega_i) - T(\eta_M, \omega_i) \right| \\
&\quad + \left| \frac{1}{M} \sum_{i=1}^{M} T(\eta_M, \omega_i) - T^{\mathcal{N}(M)}(\eta_M, \omega_i) \right| \\
&\le |J(\eta) - J^M(\eta_M)| + L_T \|\eta - \eta_M\| + K_T/\mathcal{N}(M)
\end{aligned}
$$

The result then follows from Assumption 15 and the almost sure epiconvergence of J^M to J. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

This result establishes that Problem D_C can be approximated by a sequence of high-dimensional nonlinear programming problems by using a sample average scheme to approximate the expectation over the parameter space and an Euler scheme to discretize the time domain. The resulting numerical solutions will be meaningful in the sense that an accumulation point of a sequence of global minimizers to the approximate problem will be a global minimizer of the original problem. In order to establish a similar result for stationary points, we must develop optimality conditions for the approximate problem and analyze the approximation of the adjoint variables. Such a result is beyond the scope of this work.

4.6 Numerical Examples

In Section 4.1-4.5 we propose a computational framework for the uncertain optimal control problem and demonstrated that it can be approximated by a sequence of high-dimensional nonlinear programming problems (NLPs) under mild regularity assumptions. In this section we provide a number of example problems which demonstrate this process. Each problem is approximated numerically by taking a random sample of size M from the parameter space using a Monte Carlo method and approximating the objective functional using the sample average. The resulting standard optimal control problem is discretized using a LGL-pseudospectral method with N nodes in the time domain. This yields an MN dimensional NLP which then is solved using the sequential quadratic programming package SNOPT [27]. Proposition 2.3.1 and Theorem 4.5.2 guarantee that if the resulting sequence of approximate optimal controls converges in $\mathcal{L}_2$, the accumulation point will be the optimal solution to the original uncertain optimal control problem.

Consider the problem of designing a controller to drive a harmonic oscillator with natural frequency in the range $\omega \in [\delta_0, \delta_1]$, so as to minimize the expectation of some cost functional. The oscillator in question is modelled by the uncertain dynamical system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad (4.21)$$

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for all $\omega \in [\delta_0, \delta_1]$, $t \in [0, t_f]$. The goal of the UOCP is to minimize the cost functional

$$
\begin{aligned}
J =& \mathbb{E}^P \left[\beta \big[(x_1(t_f, \omega))^2 + (x_2(t_f, \omega))^2 \big] + \gamma \int_0^{t_f} \big[(u_1(t))^2 + (u_2(t))^2 \big] dt \right] \\
=& \beta \, \mathbb{E}^P \left[(x_1(t_f, \omega))^2 + (x_2(t_f, \omega))^2 \right] + \gamma \int_0^{t_f} \big[(u_1(t))^2 + (u_2(t))^2 \big] dt \qquad (4.22)
\end{aligned}
$$

Here β and γ are scale factors which weight the priority of minimizing the error of the final state against minimizing the expended control energy.

In this section we use the computational framework proposed in this paper to numerically calculate an optimal control for this ensemble of oscillators with or without the presence of control constraints.

Problem S_O Find a control $u : (-1000, 1000) \mapsto \mathbb{R}^2$ to minimize the objective functional 4.22 subject to the uncertain dynamical system 4.21, where $t_f = 1, \delta_0 = 0, \delta_1 = 20, \beta = 10, \gamma = 0.1$.

This problem approximates the unconstrained problem by allowing the admissable controls to take values in a large open subset of $\mathbb{R}$. The proposed computational framework of this paper is applied to this optimal ensemble control problem by taking a random uniformly distributed draw of size M from the parameter space and approximating (4.22) by the sample average. The resulting standard optimal control problem is solved using a direct method based on an LGL-pseudospectral direct discretization scheme with 54 nodes in the time domain. A sample computed trajectory for $M = 26$ is shown in Figure 4.1. The optimal control and a sample of final states for the optimal trajectory

is shown in Figure4.2.

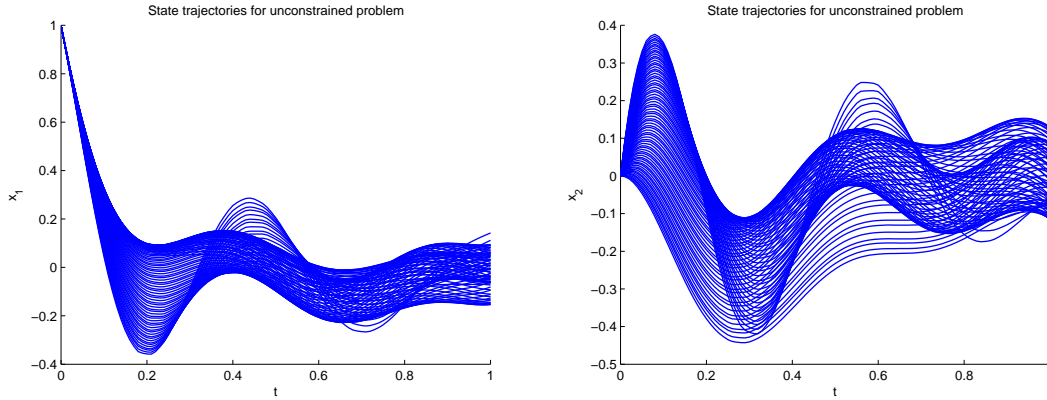The antisymmetry of the state dynamics and quadratic form of the cost functional

Figure 4.1: A sample of state trajectories for a controlled ensemble of harmonic oscillators with variation in the natural frequency. Here the objective is to minimize a linear combination of the expectation norm of the final state and the controlled energy expended, with no constraint on the control. The optimal control is computed the sample average scheme introduced in this chapter and an LGL quadrature scheme in the parameter space, along with the NLP package SNOPT.

allow this problem an easily verifiable necessary condition. First we cast the problem

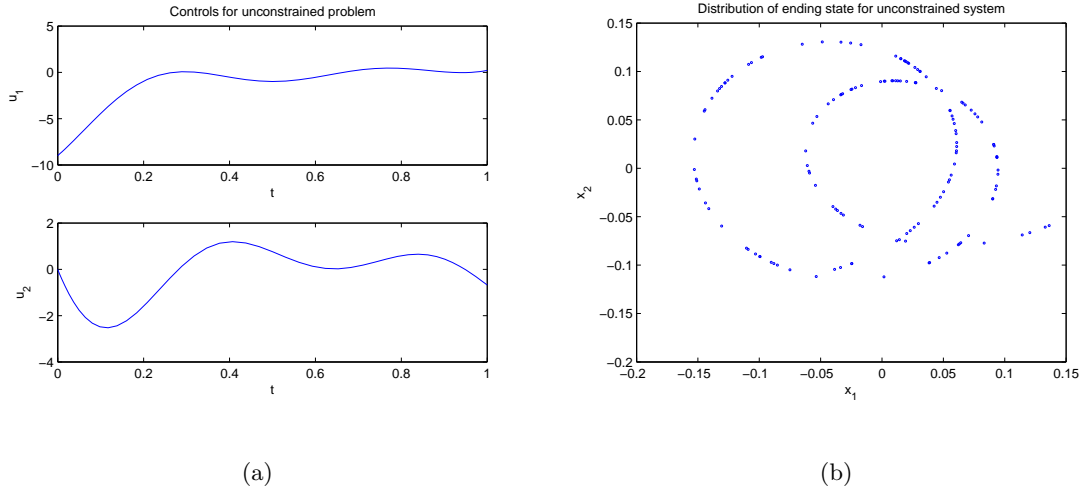in the form of Section 4.1. We introduce the auxiliary state x_3 and define the state

Figure 4.2: a) The optimal control for the ensemble of simple harmonic oscillators problem with no control constraint, calculated using a sample average approximation. b) A sample of final states for a controlled ensemble of harmonic oscillators with variation in the natural frequency.

dynamics by

$$
\begin{bmatrix} \dot{x}_1(t,\omega) \\ \dot{x}_2(t,\omega) \\ \dot{x}_3(t,\omega) \end{bmatrix} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t,\omega) \\ x_2(t,\omega) \\ x_3(t,\omega) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \\ \gamma(u_1(t))^2 + (u_2(t))^2 \end{bmatrix},
$$
$$
\begin{bmatrix} x_1(0,\omega) \\ x_2(0,\omega) \\ x_3(0,\omega) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{4.23}
$$

The cost functional is then given by

$$
J = \mathbb{E}^P[F(x(1,\omega),\omega)] \qquad\qquad F(x,\omega) = \beta x_1^2 + \beta x_2^2 + \gamma x_3. \tag{4.24}
$$

Finally, the adjoint equation defined in (4.6) is given by

$$
\begin{bmatrix} \dot{p}_1(t,\omega) \\ \dot{p}_2(t,\omega) \\ \dot{p}_3(t,\omega) \end{bmatrix} = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1(t,\omega) \\ p_2(t,\omega) \\ p_3(t,\omega) \end{bmatrix}, \quad \begin{bmatrix} p_1(1,\omega) \\ p_2(1,\omega) \\ p_3(1,\omega) \end{bmatrix} = \begin{bmatrix} 2\beta x_1(1,\omega) \\ 2\beta x_2(1,\omega) \\ \gamma \end{bmatrix}. \tag{4.25}
$$

The necessary condition defined in Section 4.3 then requires that for an optimal solution

we have

$$
\theta_O^M(\eta) = \|\nabla J(\eta)\|_2 = 0, \tag{4.26}
$$

where ∇J is the Frechet derivative given by

$$
\begin{aligned}
\nabla J(t) &= \mathbb{E}^P[f_u^T(x(t,\omega), u(t), \omega)p(t,\omega)] \\
&= \mathbb{E}^P \begin{bmatrix} p_1(t,\omega) + 2\gamma u_1(t) \\ p_2(t,\omega) + 2\gamma u_2(t) \end{bmatrix} \\
&= \begin{bmatrix} 2\gamma u_1(t) \\ 2\gamma u_2(t) \end{bmatrix} + \mathbb{E}^P \begin{bmatrix} p_1(t,\omega) \\ p_2(t,\omega) \end{bmatrix}.
\end{aligned} \tag{4.27}
$$

The objective value and optimality function values, for a given sample size is shown in

Figure 4.3.

Problem S_C Find a control $u : [-1,1] \mapsto \mathbb{R}^2$ to minimize the objective functional

4.22 subject to the uncertain dynamical system 4.21, where $t_f = \pi, \delta_0 = 0, \delta_1 = 3, \beta = 1, \gamma = 0$.

In the constrained problem, a smaller range of natural frequencies is used because limi-

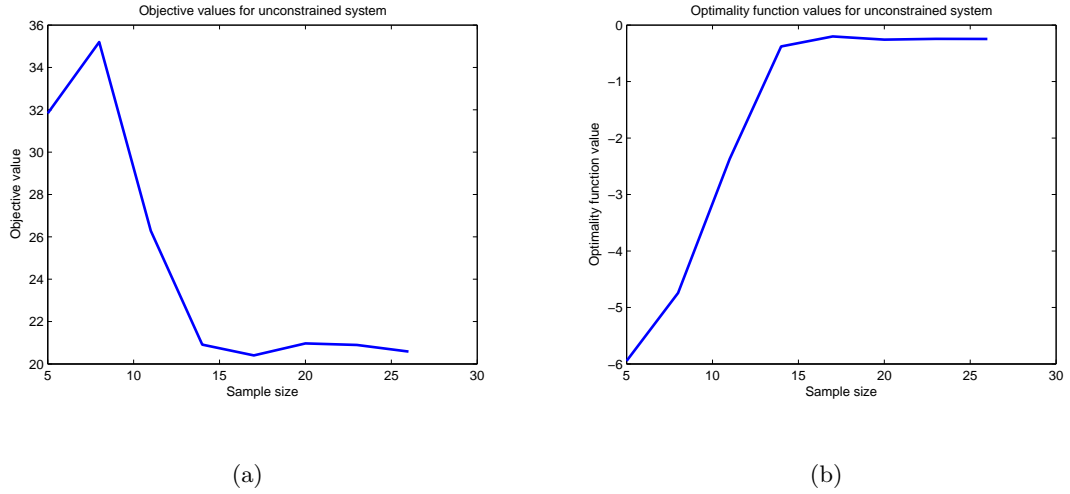(a)   (b)

Figure 4.3: a) The value of the objective functional (4.22) for Problem S_O computed using sample averages, as a function of the sample size M. b) The value of the optimality function (4.26) as a function of the sample size M.

tations on the control input make it more difficult to stabilize the system. As with the unconstrained problem, the optimal control is computed numerically using the framework proposed in this paper with an LGL-pseudospectral discretization with 36 nodes in the time domain. A sample of computed state trajectories for $M = 74$ is shown in Figure 4.4.

In Section 4.3 it is shown that an optimal solution must satisfy the necessary condition $\theta_C(\eta) = 0$, where θ_C is given by (4.9). By substituting (4.27) we have

$$
\begin{aligned}
\theta_C(u) &= \min_{u'(t)\in[-1,1]} \langle \nabla J, u' - u \rangle_2 + \frac{1}{2}\left\| u' - u \right\|_2^2 \\
&= -\langle \nabla J, u \rangle_2 + \frac{1}{2}\left\| u \right\|_2^2 + \min_{u'(t)\in[-1,1]} \langle \nabla J - u, u' \rangle + \frac{1}{2}\left\| u' \right\|_2^2
\end{aligned}
\tag{4.28}
$$

The value of the objective functional $J(u_M^*)$ and optimality function $\theta_C(u_M^*)$ for a number of sample sizes M is shown in Figure 4.6. The state variables $x(t,\omega)$ and adjoint
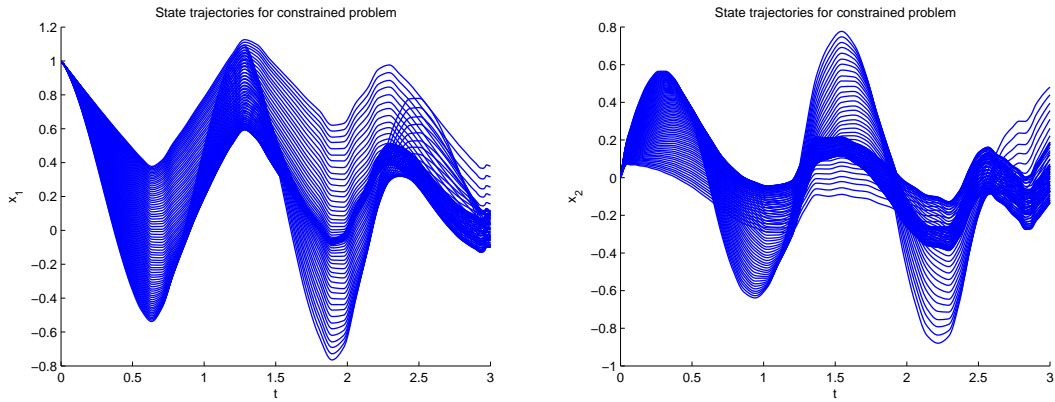
Figure 4.4: A sample of state trajectories for a controlled ensemble of harmonic oscillators with variation in the natural frequency. Here the objective is to minimize expectation of the norm of the final state subject to a pointwise control constraint. The optimal control is computed the sample average scheme introduced in this chapter and an LGL quadrature scheme in the parameter space, along with the NLP package SNOPT.

variables $p(t, \omega)$ are calculated using 54 LGL-pseudospectral nodes in the parameter space and solving the resulting state-adjoint system using the MATLAB's differential equation package *ode45*. The value of θ_C is then determined using MATLAB's quadratic programming package *quadprog*.
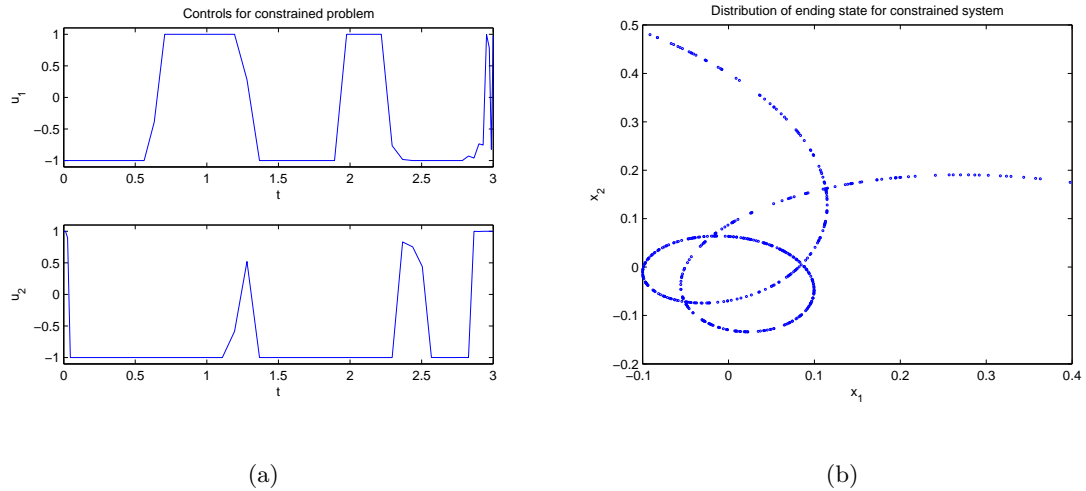
Figure 4.5: a) The optimal control for the ensemble of simple harmonic oscillators problem with a pointwise control constrained, calculated using a sample average approximation. b) A sample of final states for a controlled ensemble of harmonic oscillators with variation in the natural frequency.
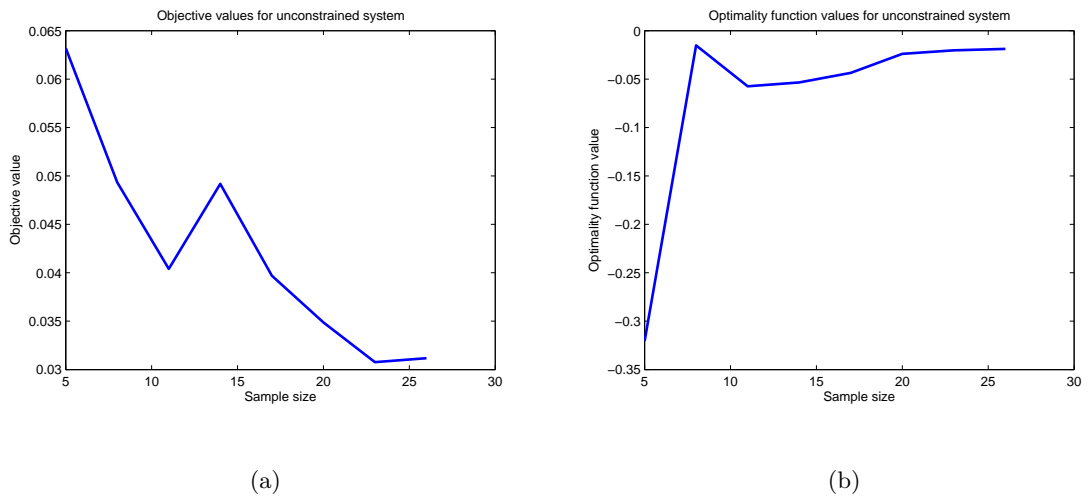

Figure 4.6: The value of the objective functional (4.22) for Problem S_C computed using sample averages, as a function of the sample size M. b) The value of the optimality function (4.28) as a function of the sample size M.

Chapter 5

Conclusion

In this thesis we focus providing computational framework for the solution of a class of optimal control which incorporate parameter uncertainty into the dynamics, objective functional, and initial states. This class of problems is inspired by a number of recently considered applications in optimal control for which the parameter uncertainty is inherent, such as optimal search and ensemble control. In addition, many existing applications of optimal control, such as trajectory optimization, can be extended in this framework to include uncertainty about physical parameters in the dynamical system or the external environment.

We provide a framework for the numerical solution of this class of problems based on a discretization of the space of stochastic parameters. In this method, the uncertain dynamical system is approximated by selecting a finite number from the parameter space using a numerical integration scheme, and the objective functional is approximated by a finite sum. The advantage of this approach is that the discretized

problem is a standard nonlinear optimal control problem which can be solved using existing methods from computational optimal control.

A rigorous analysis of convergence properties is provided for such a framework. The consistency of the proposed framework is theoretically guaranteed under mild regularity type of conditions for either a quadrature or Monte Carlo sampling schemes. In addition, we provide two types of necessary conditions for the uncertain optimal control problem which can be used for validation and verification of numerically computed solutions. First, a Pontryagin-like Hamiltonian minimization criterion is derived by analyzing the convergence properties of the dual variables for the approximate problem. This necessary condition shows that the optimal solution to the uncertain optimal control problem must minimize the expectation of the Hamiltonian over the space of stochastic parameters. Second, we provide a necessary condition based on the $\mathcal{L}_2$-Frechet derivative of objective functional in the form of an optimality function. By analyzing the convergence properties of the optimality functions we demonstrate that the numerical scheme based on sample average approximation is consistent in the sense of Polak [61, Chapter 4].

The computational framework as well as theoretical analysis on consistency can be extended in several directions which could provide interesting topics for future research. In the following, we briefly list some of these possible extensions.

- **Explicit final conditions:** In this thesis, we do not consider problems with fixed end points. Requirement on the final states is implicitly addressed by augmenting it into cost functional as a penalty term. However controllability results for

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the ensemble control problem demonstrate that for certain control problems, it is possible to guarantee the existence of an open loop control to drive the state trajectories with uncertainty into a given neighborhood of a desired point. For such problems it may be desirable to restrict the decision space for the optimization problem to only those controls which transfer the system to the desired end state. The computational methods presented in this thesis can be directly implemented on such problems, however, the consistency properties need to be carefully analyzed.

- **Free end-time problems:** In this thesis we consider problems with only fixed end times, which excludes a large number of control applications such as minimum time problems. It is possible to handle the free end-time by simply projecting the time domain $[t_0, t_f]$ to a fixed computational domain and incorporating the end-time, t_f, into the optimization decision variables. However, the challenge is that free end-time problems typically associates to explicit conditions on the final state, which introduces controllability issues. For general nonlinear dynamics, such controllability results are largely missing.

- **State constraints:** The results presented in this thesis can potentially be extended to problems with pointwise state constraints, for example, obstacle avoidance type of constraints encountered in motion planning. However, this extension presents unique challenges in both how to formulate the constraints and how to determine a control which would satisfy such state constraints for the uncertain

problem. For instance, the constraint could be formulated so as to keep the probability of collision with the obstacle within reasonable bounds. However, the inclusion of such a constraint poses new difficulties in how to formulate necessary conditions for the problem.

- **Improve the efficiency of NLP:** When the method presented in this thesis is applied to an uncertain optimal control problem, the result is a high-dimensional nonlinear programming problem (NLP). However, this NLP is highly structured, as the same time discretization step is applied to every node in the parameter space, and the discretized dynamical systems are coupled only through the open loop control. It may be possible to explore this structure in the numerical optimization algorithms, thus leading to improved performance for the solution of the uncertain optimal control problem.

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